On the multi-dimensional Favard Lemma

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The classical Favard Lemma consists to associate to any probability measure $\mu$ on the real line with finite moments of any order, two sequences called Jacobi sequences of $\mu$,

$$
((w_n)_{n \in \mathbb{N}}, (\alpha_n)_{n \in \mathbb{N}}), \quad w_n \in \mathbb{R}_+, \alpha_n \in \mathbb{R}
$$

such that

$$w_n = 0 \Rightarrow w_{n+k} = 0, \forall k \in \mathbb{N}$$

and conversely, given two such sequences, it is associated:

(i) a state, induced by a probability measure on $\mathbb{R}$, on the one indeterminate polynomial algebra $\mathcal{P}$,

(ii) an orthogonal decomposition of $\mathcal{P}$ canonically associated to this state.
The formulations of these results, in the multi-dimensional case, are recently given by identifying the theory of multi-dimensional orthogonal polynomials with the theory of symmetric interacting Fock spaces. The multi-dimensional analogue of positive numbers $w_n$ (resp. real numbers $\alpha_n$) are the positive definite matrices $(\Omega_n)_n$ (resp. Hermitean matrices $\alpha_n$) such that $\Omega_n \in \mathcal{B}((\mathbb{C}^d)^\otimes n)$ and $\alpha_n : \nu \in \mathbb{C}^d \mapsto \alpha_{\nu|n} \in \mathcal{B}((\mathbb{C}^d)^\otimes n)$. 

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Let \( d \in \mathbb{N}^* \) and let

\[ P = \mathbb{C}[(X_j)_{1 \leq j \leq d}] \]

be the complex polynomial algebra in the commuting indeterminates \((X_j)_{1 \leq j \leq d}\) with the \(*\)-structure uniquely determined by the prescription that the \(X_j\) are self-adjoint with respect to a pre-scalar product induced by a state \(\varphi\). Then, it is proved by Accardi-Barhoumi-Dhahri that there exist:
1) an orthogonal decomposition of $\mathcal{P}$

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n$$

3) three operators $a_{j|n}^\varepsilon$, $\varepsilon \in \{+ , 0 , -\}$, defined with respect to a basis $e = (e_j)_{1 \leq j \leq d}$ of $\mathbb{C}^d$ as follows:

$$a_{j|n}^+ = a_{e_j|n}^+ : \mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$$

$$a_{j|n}^0 = a_{e_j|n}^0 : \mathcal{P}_n \rightarrow \mathcal{P}_n$$

$$a_{j|n}^- = a_{e_j|n}^- : \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}$$

such that
the sum is orthogonal and meant in the weak sense, i.e. for each element \( Q \in \mathcal{P} \) there is a finite set \( I \subset \mathbb{N} \) such that

\[
Q = \sum_{n \in I} p_n, \quad p_n \in \mathcal{P}_n
\]  

Moreover

\[
a_j^+ := \sum_{n \in \mathbb{N}} a_{j|n}^+, \quad a_j^0 := \sum_{n \in \mathbb{N}} a_{j|n}^0, \quad a_j^- := \sum_{n \in \mathbb{N}} a_{j|n}^-
\]

\[
(a_{j|n}^+)^* = a_{j|n+1}^-; \quad (a_j^+)^* = a^-_j
\]

\[
(a_{j|n}^0)^* = a_{j|n}^0; \quad (a_j^0)^* = a_j^0
\]

Moreover

\[
X_j = a_j^+ + a_j^0 + a_j^-
\]
Notation: If $v = (v_1, \ldots, v_d) \in \mathbb{C}^d$, where $v_1, \ldots, v_d$ are the coordinates of $v$ in the basis $e$, we denote

$$a_v^e|_n := \sum_{1 \leq j \leq d} v_j a_j^e|_n$$

It is proved by Accardi-Barhoumi-Dhahri that:

1) $P_{n+1}$ is generated by the set $\{a_v^+(P_n), \ v \in \mathbb{C}^d\}$ and $[a_v^+, a_w^+] = 0$ for all $v, w \in \mathbb{C}^d$.

2) there exist a unitary operator

$$U_n: v_n \hat{\otimes} v_{n-1} \hat{\otimes} \cdots \hat{\otimes} v_1 \in (\mathbb{C}^d)^{\hat{\otimes} n} \rightarrow a_v^+ a_{v_{n-1}}^+ \cdots a_{v_1}^+ \Phi \in P_n,$$

where $\Phi = 1_P$.

3) $\alpha.|_n = U_n^{-1} a_0^0|_n U_n$ and $\Omega_n$ is defined by the following:

$$\langle U_n \xi_n, U_n \eta_n \rangle_n := \langle \xi_n, \Omega_n \eta_n \rangle_{(\mathbb{C}^d)^{\hat{\otimes} n}}; \quad \xi_n, \eta_n \in (\mathbb{C}^d)^{\hat{\otimes} n} \quad (2)$$
Proposition

If \( v \neq 0 \in \mathbb{C}^d \), the creator operator \( a_v^+ \) is an injective operator, i.e:

\[
a_v^+ \xi_n = 0 \Rightarrow \xi_n = 0
\]

Corollary

If \( v_1, \ldots, v_n \) are \( n \)-linearly independent vectors of \( \mathbb{C}^d \), then \( a_{v_1}^+, \ldots, a_{v_n}^+ \) are linearly independent.
Denote $\langle \cdot, \cdot \rangle_n$ the pre-scalar product induced by a state $\varphi$ on $\mathcal{P}_n$.

1. First case:
Let us fix a linear basis $e = (e_j)_{1 \leq j \leq d}$ of $\mathbb{C}^d$ and suppose that the CAP operators are defined with respect to the basis $e$, i.e.:

$$a_j^\varepsilon = a_{e_j}^\varepsilon$$

where $\varepsilon \in \{+, 0, -\}$. 
Recall that

1) \[ a_v^+ = \sum_{1 \leq j \leq d} \alpha_j a_j^+ \] (3)

where \( v = \alpha_1 e_1 + \cdots + \alpha_d e_d \).

2) \[ X_v = \sum_{1 \leq j \leq d} \alpha_j X_j \]

Let \( e' = (e'_j)_{1 \leq j \leq d} \) be another basis of \( \mathbb{C}^d \). Denote

3) \[ b_j^+ = a_{e'_j}^+ \] (4)

4) \[ b_v^+ = \sum_{1 \leq j \leq d} \beta_j b_j^+ \], where \( v = \beta_1 e'_1 + \cdots + \beta_d e'_d \)
Note that from (2), one has

\[ \langle v_1 \hat{\otimes} \ldots \hat{\otimes} v_n, w_1 \hat{\otimes} \ldots \hat{\otimes} w_n \rangle_{n,e} = \langle a^+_v \ldots a^+_v \Phi, a^+_w \ldots a^+_w \Phi \rangle_n \]

\[ = \langle v_n \hat{\otimes} \ldots \hat{\otimes} v_1, \Omega_n^{(e)} w_n \hat{\otimes} \ldots \hat{\otimes} w_1 \rangle_{(\mathbb{C}^d)^{\hat{\otimes} n}} \]

\[ \langle v_n \hat{\otimes} \ldots \hat{\otimes} v_1, w_n \hat{\otimes} \ldots \hat{\otimes} w_1 \rangle_{n,e} = \langle b^+_v \ldots b^+_v \Phi, b^+_w \ldots b^+_w \Phi \rangle_n \]

\[ = \langle v_n \hat{\otimes} \ldots \hat{\otimes} v_1, \Omega_n^{(e')} w_n \hat{\otimes} \ldots \hat{\otimes} w_1 \rangle_{(\mathbb{C}^d)^{\hat{\otimes} n}} \]

for all \( v_j, w_j \in \mathbb{C}^d \ (1 \leq j \leq n) \), where \( \langle \cdot, \cdot \rangle_{(\mathbb{C}^d)^{\hat{\otimes} n}} \) is a pre-scalar product on \( (\mathbb{C}^d)^{\hat{\otimes} n} \). Moreover, one has

\[ \alpha^{(e)}_{j|n} = \alpha^{(e')}_{e_j|n} = \alpha^{(e)}_{a_j|n} = U_n^{-1} a^0_j U_n \]

for all \( j \in \{1, \ldots, d\} \).
Theorem

For all \( n \in \mathbb{N}^* \), we have

\[
\Omega^{(e)}_n = \Omega^{(e')}_{n}, \quad \alpha^{(e')}_{j|_n} = \sum_{i=1}^{d} r_{ij} \alpha^{(e)}_{i|_n}
\]

for all \( j \in \{1, \ldots, d\} \) where \( R = (r_{ij})_{1 \leq i, j \leq d} = Pass(e, e') \).
**Second case:**
Statement of the problem: Let \( e = (e_j)_{1 \leq j \leq d} \) and \( e' = (e'_j)_{1 \leq j \leq d} \) be two basis of \( \mathbb{C}^d \). Suppose that there are two different initial choices of the basis of \( \mathbb{C}^d \) associated to the CAP operators:

1) The first choice is to define the CAP operators with respect to the basis \( e \), i.e:

\[
\alpha^e_{j} := \alpha^e_{e_j} \quad (5)
\]

2) The second choice is to define the CAP operators with respect to the basis \( e' \), i.e:

\[
\alpha^e_{j} := \alpha^e_{e'_j} \quad (6)
\]

What are the relations between the corresponding Jacobi sequences \( \Omega_n^{(e)} \) and \( \Omega_n^{(e')} \), \( \alpha^{(e')}_{j|n} \) and \( \alpha^{(e')}_{j|n} \)?
Theorem

Let $\Omega^{(e)}_n$ and $\Omega^{(e')}_n$ be the positive definite matrices associated to the two choices 1) and 2). Let $R = (r_{ij})_{1 \leq i, j \leq d} = \text{Pass}(e, e')$. Then for all $j \in \{1, \ldots, d\}$

\[
\Omega^{(e)}_n = (R^\otimes n)^* \Omega^{(e')}_n R^\otimes n = (R^*)^\otimes n \Omega^{(e')}_n R^\otimes n
\]

\[
\alpha^{(e')}_{j|n} = \sum_{i=1}^{d} r_{ij} \alpha^{(e)}_{i|n}
\]

Moreover, if $e$ and $e'$ are orthonormal basis of $\mathbb{C}^d$, then

\[
\Omega^{(e)}_n = (R^{-1})^\otimes n \Omega^{(e')}_n R^\otimes n
\]
The connection between the classical and multi-dimensional Favard Lemmas in the case of $d = 1$

Let $d = 1$, $\mu$ be a probability measure on $\mathbb{R}$ and $(\alpha_n, w_n, P_n)_n$ be the classical Favard Lemma sequences associated to $\mu$. Then the Jacobi relation is given by

$$XP_n = w_nP_{n+1} + \alpha_nP_n + w_{n-1}P_{n-1}, \quad P_{-1} = 0$$

We define the CAP operators as follows:

$$a_1^+|_nP_n = w_nP_{n+1}$$
$$a_1^-|_nP_n = w_{n-1}P_{n-1}$$
$$a_1^0|_nP_n = \alpha_nP_n$$
It is clear that \((a^0_1|_n)^* = a^0_1|_n\) and \(a^-_1|_n = (a^+_1|_{n-1})^*\) with respect to the pre-scalar product induced by \(\mu\) on \(\mathcal{P}_n\). Note that if \(a^+_1\) is defined with respect to two different choices of basis \(e = (e_1)\) and \(e' = (e'_1)\) of \(\mathbb{C}\) \((d = 1)\):

\[
\begin{align*}
a^+_1 &= a^+_e e_1 \\
a^+_1 &= a^+_{e'_1}
\end{align*}
\]

and if we denote by \(R = Pass(e, e')\) (i.e \(Re_1 = e'_1 = ae_1\)), then the positive scalars \(\Omega^{(e)}_n\) and \(\Omega^{(e')}_n\) (because \(d = 1\)) satisfy

\[
\Omega^{(e)}_n = (R^*)^n \Omega^{(e')}_n R^n
\]

But \(R^* v = \bar{a} v\). This gives

\[
\Omega^{(e)}_n = |a|^{2n} \Omega^{(e')}_n
\]
Now, suppose that $a_1^+ = a_{e_1}^+$ where $e = (e_1)$ is the canonical basis of $\mathbb{C}$ and denote $\Omega^{(e)}_n = \Omega_n$. A straightforward computation shows that

$$\alpha_n e_1 = \alpha_n, \quad \Omega_n = w_0^2 \ldots w_{n-1}^2$$

Note that $\Omega_n$ depends on the choice of the scalar product of $(\mathbb{C})^\otimes n$. Therefore if we choose

$$\langle \cdot, \cdot \rangle_{\mathbb{C}^\otimes n} := \frac{w_n}{w_0^2 \ldots w_{n-1}^2} \langle \cdot, \cdot \rangle_{n,c}$$

one finds in this case $\Omega_n = w_n$. 

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Let $\mu_1, \ldots, \mu_d$ be $d$ probability measures on $\mathbb{R}$. By the classical Favard Lemma there exist sequences $(\alpha_{k,n}, w_{k,n}, P_{k,n})_n$ $(1 \leq k \leq d)$ such that:

(i) $(w_{k,n})_n$ is a sequence of positive real scalars,

(ii) $(\alpha_{k,n})_n$ is a sequence of real scalars,

(iii) $(P_{k,n})_n$ is a family of orthonormal polynomials with respect to the pre-scalar product induced by the probability measure $\mu_k$ which satisfies:

$$XP_{k,n} = w_{k,n}P_{k,n+1} + \alpha_{k,n}P_{k,n} + w_{k,n-1}P_{k,n-1}$$

$$P_{k,-1} = 0$$
It is clear that

\[ P_{\vec{n}} = P_{1,n_1} \otimes \cdots \otimes P_{j,n_j} \otimes \cdots \otimes P_{d,n_d}, \quad (7) \]

where \( \vec{n} = (n_1, \ldots, n_d) \) and \( j \) indicates the \( j-th \) variable \( X_j \), is a family of orthogonal polynomials with respect to the pre-scalar product induced by \( \mu = \mu_1 \otimes \cdots \otimes \mu_d \).

Define the CAP operators with respect to the canonical basis \((e_j)_{1 \leq j \leq d} \) of \( \mathbb{C}^d \) as follows:

\[
\begin{align*}
  a^+_{k,n} P_{k,n} &= a^+_k|n P_{k,n} = a^+_e k|n P_{k,n} = w_n P_{k,n+1} \\
  a^-_{k,n} P_{k,n} &= a^-_k|n P_{k,n} = a^-_e k|n P_{k,n} = w_{n-1} P_{k,n-1} \\
  a^0_{k,n} P_{k,n} &= a^0_k|n P_{k,n} = a^0_e k|n P_{k,n} = \alpha_n P_{k,n}
\end{align*} \quad (8)
\]

The CAP operators (8) act on the tensor product (7) as follows:

\[ a^\varepsilon_{k,n} = I \otimes \cdots \otimes a^\varepsilon_{k,n} \otimes I \cdots \otimes I \]
Define the binaire relation $\mathcal{R}$ on $\{1, \ldots, d\}^n$ by
\[(i_1, \ldots, i_n)\mathcal{R}(j_1, \ldots, j_n)\]
if and only if
\[\{i_1, \ldots, i_n\} = \{j_1, \ldots, j_n\}\]
and
\[\#(\{i_k = l, k = 1, \ldots, n\}) = \#(\{j_k = l, k = 1, \ldots, n\})\]
for all $l \in \{1, \ldots, d\}$.

**Lemma**

$\mathcal{R}$ is an equivalence relation on $\{1, \ldots, d\}^n$.

Define
\[\mathcal{A}_n := \{\overline{j_n} = cl((j_1, \ldots, j_n)); j_k \in \{1, \ldots, d\}\}\]
\[e_{j_n} := e_{j_1} \hat{\otimes} \ldots \hat{\otimes} e_{j_n}\]

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Lemma

The family \( \mathcal{B} = (e_{j_n})_{j_n \in A_n} \) is a basis of \( (\mathbb{C}^d)^{\otimes n} \).

Hence \( \Omega_n \) is of the form

\[
\Omega_n = (\beta_{i_n j_n})_{i_n, j_n \in A_n}
\]
Theorem

For all $\bar{i}_n = cl(i_1, \ldots, i_n); \bar{j}_n = cl(j_1, \ldots, j_n) \in A_n$ we have

$$\beta_{\bar{i}_n \bar{j}_n} = \delta_{\bar{i}_n \bar{j}_n} \left( \prod_{k_1=0}^{m_1-1} w_{1,k_1} \right) \cdots \left( \prod_{k_r=0}^{m_r-1} w_{r,k_r} \right) \cdots \left( \prod_{k_d=0}^{m_d-1} w_{d,k_d} \right)$$

$$\alpha_{e_l|_n} e_{\bar{i}_n} = \alpha_{l,m_l} e_{\bar{i}_n}$$

where $m_l = \# \left( \{ i_k = l; k = 1, \ldots, n \} \right)$ (1 ≤ l ≤ d) with the convention $\prod_{k=0}^{-1} w_{l,k} = 1$ (this convention is used when $m_l = 0$).