Model Uncertainty and Robust Duality in Finance

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Introduction

In the latest years, partly due to the financial crisis, there has been an increased focus on *model uncertainty* in mathematical finance. Here model uncertainty is understood in the sense of uncertainty with respect to the choice of the underlying probability measure. This is sometimes called *Knightian uncertainty*, after the University of Chicago economist Frank Knight (1885-1972). The corresponding model uncertainty stochastic control problem is often called *robust control*.

Optimizing under model uncertainty is also of interest because it is related to *risk minimization* problems, where risk is interpreted in the setting of a *convex risk measure* (Föllmer & Schied (2002), Frittelli & Rosazza-Gianin (2002)).
Outline

- Duality method in portfolio optimization; use of duality relationships in the space of convex functions and semimartingales.
- Itô - Lévy market case
- Maximum principles for stochastic control
- Optimal portfolio and optimal scenario
- Model uncertainty and robust control
- Extension to robust duality
- Illustrating examples
Financial market

Let $S(t); \ 0 \leq t \leq T$ represent discounted unit price of a risky asset at time $t$. We assume that $S(t)$ is a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$.

$T$ is a finite time horizon.

Let $\varphi(t)$ be an $\mathcal{F}_t$-predictable $S$-integrable portfolio process, giving the number of units held of the risky asset at time $t$. If $\varphi(t)$ is self-financing, the corresponding wealth process $X(t) = X^x_\varphi(t)$ is given by

$$X(t) = x + \int_0^t \varphi(s) dS(s); \ 0 \leq t \leq T,$$

(0.1)

where $x \geq 0$ is the initial value of the wealth. We say that $\varphi$ is admissible and write $\varphi \in A$ if $X_\varphi(t) \geq 0$ for all $t \in [0, T]$ a.s.
Let $\mathcal{M}$ be the set of probability measures $Q$ which are equivalent local martingale measures (ELMM), in the sense that $Q \sim P$ and $S(t)$ is a local martingale under $Q$. We assume that

$$\mathcal{M} \neq \emptyset$$

which is related to absence of arbitrage opportunities on the security market.
Let $U : [0, \infty] \to \mathbb{R}$ be a utility function, strictly increasing, strictly concave, $C^1$ and satisfying the Inada conditions:

$$U'(0) = \lim_{x \to 0^+} U'(x) = \infty \quad U'(\infty) = \lim_{x \to \infty} U'(x) = 0.$$ 

Let $V$ be the conjugate function of $U$:

$$V(y) := \sup_{x>0} \{ U(x) - xy \} ; \ y > 0$$

$V$ is strictly convex, decreasing, $C^1$ and satisfies

$$V'(0) = -\infty, \ V'(\infty) = 0, \ V(0) = U(\infty), \ \text{and} \ V(\infty) = U(0).$$

Moreover,

$$U(x) = \inf_{y>0} \{ V(y) + xy \} ; \ x > 0, \ \text{and}$$

$$U'(x) = y \iff x = -V'(y).$$

i.e. $U'$ is the inverse function of $-V'$. 
Duality approach for optimal portfolio problem

The *primal problem* is to find \( \varphi^* \in A \) such that

\[
u(x) := \sup_{\varphi \in A} E[U(X^x_\varphi(T))] = E[U(X^x_{\varphi^*}(T))].
\] (0.2)

The *dual problem* to (0.2) is for given \( y > 0 \) to find \( Q^* \in \mathcal{M} \) s.t.

\[
v(y) = \inf_{Q \in \mathcal{M}} E \left[ V \left( y \frac{dQ}{dP} \right) \right] = E \left[ V \left( y \frac{dQ^*}{dP} \right) \right].
\] (0.3)

Kramkov and Schachermayer (2003) prove that, under some conditions, \( \varphi^* \) and \( Q^* \) exist and are related by

\[
U'(X^x_{\varphi^*}(T)) = y \frac{dQ^*}{dP} \quad \text{with } y = u'(x)
\] (0.4)

i.e.

\[
X^x_{\varphi^*}(T) = -V' \left( y \frac{dQ^*}{dP} \right) \quad \text{with } x = -v'(y).
\] (0.5)

We extend this result by using stochastic control theory when the risky asset price \( S(t) \) is an Itô-Lévy process.
Itô-Lévy market

We consider a financial market model, where the discounted unit price $S(t)$ of the risky asset is given by a jump diffusion of the form

$$
\begin{align*}
\left\{ 
\begin{array}{l}
    dS(t) = S(t^-) \left( b_t dt + \sigma_t dB_t + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right) ; 0 \leq t \leq T \\
    S(0) > 0
\end{array}
\right.
\end{align*}
$$

(0.6)

where $b_t, \sigma_t$ and $\gamma(t, \zeta)$ are predictable processes satisfying $\gamma(t, \zeta) > -1$ and

$$
E \left[ \int_0^T \left\{ |b_t| + \sigma_t^2 + \int_{\mathbb{R}} \gamma^2(t, \zeta) \nu(d\zeta) \right\} dt \right] < \infty. 
$$

(0.7)

Here $B(t)$ is a Brownian motion and $\tilde{N}(dt, d\zeta) := N(dt, d\zeta) - \nu(d\zeta)dt$ is an independent compensated Poisson random measure, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, $P$ is a reference probability measure and $\nu$ is the Lévy measure of $N$. 
Primal Problem

Let \( \varphi(t) \) be a self-financing portfolio, giving the number of units held of the risky asset at time \( t \) and let \( X(t) = X_\varphi(t) \) be the corresponding wealth process given by

\[
\begin{align*}
\left\{
\begin{array}{l}
dX(t) = \varphi(t)S(t^-) \left[ b_t dt + \sigma_t dB_t + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta) \right] ; \\ X(0) = x > 0.
\end{array}
\right.
\end{align*}
\]

We say that \( \varphi \) is admissible and write \( \varphi \in A \) if

1. \( \varphi(t) \) is an \( \mathcal{F}_t \)-predictable \( S \)-integrable process
2. \( X(t) > 0 \) for all \( t \in [0, T] \) a.s.
3. \( E \left[ \int_0^T \varphi(t)^2 S(t)^2 \left\{ b_t^2 + \sigma_t^2 + \int_{\mathbb{R}} \gamma^2(t, \zeta) \nu(d\zeta) \right\} dt \right] < \infty. \)
4. \( \exists \epsilon > 0 \) s.t. \( E[\int_0^T |X(t)|^{2+\epsilon} dt] < \infty \) and
5. \( E[U'(X(T))^{2+\epsilon}] < \infty. \)
Primal problem: Find $\varphi^* \in \mathcal{A}$ such that

$$u(x) := \sup_{\varphi \in \mathcal{A}} E[U(X_\varphi^x(T))] = E[U(X_{\varphi^*}^x(T))].$$  \hspace{1cm} (0.8)
Dual Problem

We represent the set \( \mathcal{M} \) of ELMM by the family of positive measures \( Q_\theta \) of the form

\[
dQ_\theta(\omega) = G_\theta(T)dP(\omega) \quad \text{on } \mathcal{F}_T,
\]

where

\[
\begin{cases}
dG_\theta(t) = G_\theta(t^-) \left[ \theta_0(t)dB_t + \int_{\mathbb{R}} \theta_1(t, \zeta)\tilde{N}(dt, d\zeta) \right] ; & 0 \leq t \leq T \\
G_\theta(0) = y > 0.
\end{cases}
\]

and \( \theta = (\theta_0, \theta_1) \) is a predictable process satisfying the conditions

\[
E \left[ \int_0^T \left\{ \theta_0^2(t) + \int_{\mathbb{R}} \theta_1^2(t, \zeta)\nu(d\zeta) \right\} dt \right] < \infty ; \quad \theta_1(t, \zeta) \geq -1;
\]

\[
b_t + \sigma_t\theta_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(t, \zeta)\nu(d\zeta) = 0 ; \quad t \in [0, T]. \quad (0.9)
\]

If \( y = 1 \), by the Girsanov theorem, this characterises \( Q_\theta \) as an ELMM.
Let $\Theta$ denote the set of processes $\theta$ satisfying the above conditions.

**Dual problem** : For given $y > 0$, find $\theta^*_y \in \Theta$ and $\nu(y)$ s. t.

$$- \nu(y) := \sup_{\theta \in \Theta} E[-V(G^y_\theta(T))] = E[-V(G_{\theta^*_y}(T))]. \quad (0.10)$$

We will use the maximum principle for stochastic control to study the primal problem and relate it to the dual problem.
Stochastic Control of Jump Diffusions

There are two main methods for solving stochastic control problems:

(i) Dynamic programming and the HJB equation
(R. Bellman ...) This method is very efficient when it works, but it assumes that the system is Markovian.

(ii) The maximum principle with the associated BSDE
(Pontryagin, Bismut, Bensoussan, Pardoux-Peng, Framstad-Ø. -Sulem,...) This method is more robust; it does not assume that the system is Markovian and it applies even to partial information, to systems with delay, to FBSDEs and to SPDEs. The drawback is that it involves complicated BSDEs, ABSDEs, BSPDEs, ...

Thus, in our general non-Markovian setting it is natural to use the maximum principle, and we will now use it to study the primal and dual problem above.
Maximum Principles for Stochastic Control

Consider the following general stochastic control problem:

$$\sup_{u \in A} E \left[ \int_0^T f(t, X(t), u(t), \omega) dt + \phi(X(T), \omega) \right]$$  \hspace{1cm} (0.11)

where $A$ is a family of admissible $\mathcal{F}$-predictable controls and

$$dX(t) = b(t, X(t), u(t), \omega) dt + \sigma(t, X(t), u(t), \omega) dB_t$$ \hspace{1cm} (0.12)

$$+ \int_\mathbb{R} \gamma(t, X(t), u(t), \omega, \zeta) \tilde{N}(dt, d\zeta) \; ; \; 0 \leq t \leq T; \; X(0) = x \in \mathbb{R}$$

Assume $b, \sigma, \gamma, f, \phi \in C^1$ and

$$E\left[ \int_0^T \left( |\nabla b|^2 + |\nabla \sigma|^2 + ||\nabla \gamma||^2 + |\nabla f|^2 \right) dt \right] < \infty$$
For $u \in \mathcal{A}$ we assume the solution $X^u(t)$ of (0.12) exists, is unique and satisfies, for some $\epsilon > 0$, $E[\int_0^T |X^u(t)|^{2+\epsilon} dt] < \infty$ and $E[\phi'(X^u(T))^{2+\epsilon}] < \infty$.

Let $\mathbb{U}$ be a convex closed set containing all possible control values $u(t); t \in [0, T]$.
Define the Hamiltonian $H : [0, T] \times \mathbb{R} \times \mathbb{U} \times \mathbb{R}^2 \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ by

$$H(t, x, u, p, q, r(\cdot)), \omega) := f(t, x, u, \omega) + b(t, x, u, \omega)p + \sigma(t, x, u, \omega)q$$

$$+ \int_{\mathbb{R}} \gamma(t, x, u, \zeta, \omega)r(t, \zeta)\nu(d\zeta).$$

The associated BSDE for the adjoint processes $(p, q, r)$ is

$$\begin{cases}
    dp(t) = -\frac{\partial H}{\partial x}(t)dt + q(t)dB_t + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\
p(T) = \phi'(X(T))
\end{cases} \tag{0.13}$$
Theorem (Sufficient Maximum Principle, Ø. & Sulem, JOTA 2012)

Let $\hat{u} \in A$ with corresponding solutions $\hat{X}, \hat{p}, \hat{q}, \hat{r}$ be such that

$$\sup_{v \in U} H(t, \hat{X}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) = H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)).$$

Assume

1. The function $x \mapsto \phi(x)$ is concave
2. (The Arrow condition) The function $H(x) := \sup_{v \in U} H(t, x, v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$ is concave for all $t \in [0, T]$.

Then $\hat{u}$ is an optimal control for the problem (0.11).
Theorem (Necessary Maximum Principle, Ø. & Sulem, JOTA 2012)

Assume

- For all \( t_0 \in [0, T] \) and all bounded \( \mathcal{F}_{t_0} \)-measurable random variables \( \alpha(\omega) \), the control \( \beta := 1_{[t_0, T]} \alpha \) belongs to \( \mathcal{A} \).
- For all \( u, \beta \in \mathcal{A} \) with \( \beta \) bounded, there exists \( \delta > 0 \) s.t. 
  \( \tilde{u} := u + a\beta \) belongs to \( \mathcal{A} \) for all \( a \in (-\delta, \delta) \).
- The derivative process \( x_t := \frac{d}{da} X^{u+a\beta}(t) \big|_{a=0} \), exists and belongs to \( L^2(dm \times dP) \).

Then, if \( u^* \in \mathcal{A} \) is optimal,

\[
\frac{\partial H}{\partial u}(t, X^*(t), u^*, p^*(t), q^*(t), r^*(t, \cdot)) = 0 \text{ for all } t \in [0, T].
\]
To illustrate the method, we first prove two useful auxiliary results:

**Lemma**

**Primal problem.**

Let \( \hat{\phi}(t) \in \mathcal{A} \). Then \( \hat{\phi}(t) \) is optimal for the primal problem (0.8) if and only if the (unique) solution \((\hat{p}_1, \hat{q}_1, \hat{r}_1)\) of the BSDE

\[
\begin{cases}
  d\hat{p}_1(t) = \hat{q}_1(t)dB(t) + \int_{\mathbb{R}} \hat{r}_1(t, \zeta) \tilde{N}(dt, d\zeta) ; \ 0 \leq t \leq T \\
  \hat{p}_1(T) = U'(X_{\hat{\phi}}(T)).
\end{cases}
\]

(0.14)

satisfies the equation

\[
b(t)\hat{p}_1(t) + \sigma(t)\hat{q}_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\hat{r}_1(t, \zeta)\nu(d\zeta) = 0 ; \ t \in [0, T].
\]

(0.15)
Proof. (Sketch)
(i) First assume that $\hat{\varphi} \in \mathcal{A}$ is optimal for the primal problem (0.8). Then by the necessary maximum principle, the corresponding Hamiltonian, given by

$$H_1(t, x, \varphi, p, q, r) = \varphi S(t^-)(b(t)p + \sigma q + \int_{\mathbb{R}} \gamma(t, \zeta)r(\zeta)\tilde{N}(dt, d\zeta))$$

(0.16)

satisfies

$$\frac{\partial H_1}{\partial \varphi}(t, x, \varphi, \hat{p}_1(t), \hat{q}_1(t), \hat{r}_1(t, \cdot)) \bigg|_{\varphi=\hat{\varphi}(t)} = 0,$$

where $(\hat{p}_1, \hat{q}_1, \hat{r}_1)$ satisfies (0.14) since $\frac{\partial H_1}{\partial x} = 0$. This implies (0.15).
(ii) Conversely, suppose the solution \((\hat{p}_1, \hat{q}_1, \hat{r}_1)\) of the BSDE (0.14) satisfies (0.15). Then \(\hat{\varphi}\), with the associated \((\hat{p}_1, \hat{q}_1, \hat{r}_1)\) satisfies the conditions for the sufficient maximum principle, and hence \(\hat{\varphi}\) is optimal. \qed
We now turn to the dual problem (0.10). In the following we assume that
\[ \sigma(t) \neq 0 \text{ for all } t \in [0, T]. \] (0.17)
This is more because of convenience and notational simplicity than of necessity.

**Lemma**

**Dual problem.**

Let \( \hat{\theta} \in \Theta \). Then \( \hat{\theta} \) is an optimal scenario for the dual problem (0.10) if and only if the solution \( (\hat{p}_2, \hat{q}_2, \hat{r}_2) \) of the BSDE

\[
\begin{align*}
\frac{d\hat{p}_2(t)}{\sigma(t)} &= b(t)dt + \hat{q}_2(t)dB(t) + \int_{\mathbb{R}} \hat{r}_2(t, \zeta)\tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\
\hat{p}_2(T) &= -V'(G_{\hat{\theta}}^Y(T)).
\end{align*}
\]

also satisfies

\[
-\frac{\hat{q}_2(t)}{\sigma(t)}\gamma(t, \zeta) + \hat{r}_2(t, \zeta) = 0 ; 0 \leq t \leq T.
\] (0.19)
Proof.
The Hamiltonian $H_2$ associated to (0.10) is

$$H_2(t, g, \theta_0, \theta_1, p, q, r) = g\theta_0 q + g \int_{\mathbb{R}} \theta_1(\zeta) r(\zeta) \nu(d\zeta). \quad (0.20)$$

By (0.17), the constraint (0.9) can be written

$$\theta_0(t) = \tilde{\theta}_0(t) = -\frac{1}{\sigma(t)} \left\{ b(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(t, \zeta) \nu(d\zeta) \right\} ; \ t \in [0, T]. \quad (0.21)$$
Substituting this into (0.20) we get

\[
\tilde{H}_2(t, g, \theta_1, p_2, q_2, r_2) := H_2(t, g, \tilde{\theta}_0, \theta_1, p_2, q_2, r_2)
= g \left( -\frac{q_2}{\sigma(t)} \left\{ b(t) + \int_{\mathbb{R}} \gamma(t, \zeta)\theta_1(\zeta)\nu(d\zeta) \right\} + \int_{\mathbb{R}} \theta_1(\zeta)r_2(\zeta)\nu(d\zeta) \right). 
\]

(0.22)

The equation for the adjoint processes \((p_2, q_2, r_2)\) is thus the following BSDE:

\[
\begin{cases}
    dp_2(t) = \left[ \frac{q_2(t)}{\sigma(t)} b(t) + \int_{\mathbb{R}} \theta_1(t, \zeta) \left( \frac{q_2(t)}{\sigma(t)} \gamma(t, \zeta) - r_2(t, \zeta) \right) \nu(d\zeta) \right] dt \\
    + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t, \zeta)\tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\
    p_2(T) = -V'(G_\theta(T)).
\end{cases}
\]

(0.23)
If there exists a maximiser \( \hat{\theta}_1 \) for \( \tilde{H}_2 \) then

\[
(\nabla_{\theta_1} \tilde{H}_2)_{\theta_1 = \hat{\theta}_1} = 0,
\]

(0.24)

i.e.

\[
- \frac{\hat{q}_2(t)}{\sigma(t)} \gamma(t, \zeta) + \hat{r}_2(t, \zeta) = 0 ; 0 \leq t \leq T, \quad (0.25)
\]

where \((\hat{p}_2, \hat{q}_2, \hat{r}_2)\) is the solution of (0.23) corresponding to \( \theta = \hat{\theta} \).

We thus get (0.18) and this ends the necessary part.

The sufficient part follows from the fact that the functions

\[
g \to - V(g)
\]

and

\[
g \to \sup_{\theta_1} \tilde{H}_2(t, g, \theta_1, \hat{p}_2(t), \hat{q}_2(t), \hat{r}_2(t, \cdot)) = -g \frac{\hat{q}_2(t)}{\sigma(t)} b(t)
\]

are concave. \( \square \)
We now use the above general machinery of stochastic control to obtain an explicit connection between an optimal portfolio $\hat{\varphi} \in \mathcal{A}$ for the primal problem and an optimal $\hat{\theta} \in \Theta$ for the dual problem.
Theorem 1  a) Suppose $\hat{\phi} \in \mathcal{A}$ is optimal for the primal problem

$$\sup_{\phi \in \mathcal{A}} E[U(X_\phi^x(T))].$$

Let $(p_1, q_1, r_1)$ be the associated adjoint processes, solution of the BSDE

$$\begin{cases}
 dp_1(t) = q_1(t)dB(t) + \int_{\mathbb{R}} r_1(t, \zeta)\tilde{N}(dt, d\zeta); \ 0 \leq t \leq T \\
 p_1(T) = U'(X_{\hat{\phi}}^x(T)).
\end{cases}$$

Define

$$\hat{\theta}_0(t) := \frac{q_1(t)}{p_1(t^-)}; \quad \hat{\theta}_1(t, \zeta) := \frac{r_1(t, \zeta)}{p_1(t^-)}.$$

Suppose $E[\int_0^T \{\hat{\theta}_0^2(t) + \int_{\mathbb{R}} \hat{\theta}_1^2(t, \zeta)\nu(d\zeta)\}dt] < \infty; \text{ and } \hat{\theta}_1 > -1.$
Then \( \hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1) \in \Theta \) is optimal for the dual problem

\[
\sup_{\theta \in \Theta} E[-V(G^\gamma_\theta(T))]
\]

with initial value \( y = p_1(0) \).

Moreover, the optimal density process \( G^\gamma_\theta \) starting at \( y = p_1(0) \) coincides with the optimal adjoint process \( p_1 \) for the primal problem, i.e.

\[
G^\gamma_\theta(t) = p_1(t); \quad 0 \leq t \leq T. \tag{0.26}
\]

In particular,

\[
G^\gamma_\theta(T) = U'(X^x_\hat{\phi}(T)). \tag{0.27}
\]
b) Conversely, suppose $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1) \in \Theta$ is optimal for the dual problem

$$\sup_{\theta \in \Theta} E[-V(G^\gamma_{\hat{\theta}}(T))]$$

Let $(p_2, q_2, r_2)$ be the associated adjoint processes, solution of the BSDE

$$\begin{cases}
    dp_2(t) = \frac{q_2(t)}{\sigma(t)} b(t) dt + q_2(t) dB_t + \int_{\mathbb{R}} r_2(t, \zeta) \tilde{N}(dt, d\zeta) ; 0 \leq t \leq T \\
p_2(T) = -V'(G^\gamma_{\hat{\theta}}(T)).
\end{cases}$$

Then

$$\hat{\varphi}(t) := \frac{q_2(t)}{\sigma_t S(t^-)} \mathbf{1}_{\sigma_t \neq 0} + \frac{r_2(t, \zeta)}{\gamma(t, \zeta) S(t^-)} \mathbf{1}_{\sigma_t = 0, \gamma(t, \zeta) \neq 0}$$

is an optimal portfolio (if it is admissible) for the primal problem

$$\sup_{\varphi \in \mathcal{A}} E[U(X^x_{\varphi}(T))]$$

with initial value $x = p_2(0)$. 
Moreover, the optimal wealth process $X^x_{\hat{\phi}}$ starting at $x = p_2(0)$ coincides with the optimal adjoint process $p_2$ for the dual problem, i.e.

$$X^x_{\hat{\phi}}(t) = p_2(t); \quad 0 \leq t \leq T. \quad (0.28)$$

In particular

$$X^x_{\hat{\phi}}(T) = -V'(G^y_{\hat{\theta}}(T)). \quad (0.29)$$
Example

No jumps \( (N = 0) \). Then \( \Theta \) has just one element \( \hat{\theta} \) given by

\[
\hat{\theta}(t) = -\frac{b_t}{\sigma_t}.
\]

and

\[
G^\gamma_{\hat{\theta}}(T) = y \exp(-\int_0^T \frac{b_s}{\sigma_s} dB_s - \frac{1}{2} \int_0^T \frac{b_s^2}{\sigma_s^2} ds).
\]

Therefore, by Theorem 2 b), if \( (p_2, q_2) \) is the solution of the BSDE

\[
\begin{cases}
dp_2(t) = \frac{q_2(t)}{\sigma_t} b_t dt + q_2(t) dB_t ; 0 \leq t \leq T \\
p_2(T) = -V'(G_{\hat{\theta}}(T)),
\end{cases}
\]

then \( \hat{\varphi}(t) := \frac{q_2(t)}{\sigma(t)S(t^{-})} \) is an optimal portfolio for the problem

\[
\sup_{\varphi \in A} E[U(X^x_\varphi(T))]
\]

with initial value \( x = p_2(0) \).
In particular, if \( U(x) = \ln x \), then \( V(y) = -\ln y - 1 \) and \( V'(y) = -\frac{1}{y} \). So the BSDE (0.30) becomes

\[
\begin{cases}
    dp_2(t) = \frac{q_2(t)}{\sigma_t} b_t dt + q_2(t) dB_t ; 0 \leq t \leq T \\
p_2(T) = \frac{1}{y} \exp \left( \int_0^T \frac{b_s}{\sigma_s} dB_s + \frac{1}{2} \int_0^T \frac{b_s^2}{\sigma_s^2} ds \right).
\end{cases}
\]

(0.31)

To solve this equation we try \( q_2(t) = p_2(t) \frac{b_t}{\sigma_t} \). Then

\[
dp_2(t) = p_2(t) \left[ \frac{b_t^2}{\sigma_t^2} dt + \frac{b_t}{\sigma_t} dB_t \right],
\]

which has the solution

\[
p_2(t) = \frac{1}{y} \exp \left( \int_0^t \frac{b_s}{\sigma_s} dB_s + \frac{1}{2} \int_0^t \frac{b_s^2}{\sigma_s^2} ds \right) ; 0 \leq t \leq T.
\]

(0.33)

Hence (0.31) holds.
We conclude that the optimal portfolio for primal problem with initial value $x = \frac{1}{y}$ is

$$\hat{\phi}(t) = p_2(t)\frac{b_t}{\sigma_t^2 S(t^-)}.$$  

(0.34)

Note that with this portfolio we get

$$dX_{\hat{\phi}}(t) = p_2(t)\frac{b_t}{\sigma_t^2} [b_t dt + \sigma_t dB_t]$$

$$= p_2(t) \left[ \frac{b_t^2}{\sigma_t^2} dt + \frac{b_t}{\sigma_t} dB_t \right] = dp_2(t).$$  

(0.35)
Therefore

\[
\hat{\varphi}(t) = X_{\hat{\varphi}}(t) \frac{b_t}{\sigma_t^2 S(t^-)}
\] (0.36)

which means that the optimal fraction of wealth to be placed in the risky asset is

\[
\hat{\pi}(t) = \frac{\hat{\varphi}(t) S(t^-)}{X_{\hat{\varphi}}(t)} = \frac{b_t}{\sigma_t^2},
\] (0.37)

which agrees with the classical result of Merton.
Model uncertainty setup

For a different approach to robust duality see Gushkin (2011)[2]. See also the survey in Föllmer et al (2009) [1].

To get a representation of model uncertainty, we consider a family of probability measures $R = R^\kappa \sim P$, with Radon-Nikodym derivative on $\mathcal{F}_t$ given by

$$\frac{d(R^\kappa | \mathcal{F}_t)}{d(P | \mathcal{F}_t)} = Z_t^\kappa$$

where, for $0 \leq t \leq T$, $Z_t^\kappa$ is a martingale of the form

$$dZ_t^\kappa = Z_{t-}^\kappa [\kappa_0(t) dB_t + \int_{\mathbb{R}} \kappa_1(t, \zeta) \tilde{N}(dt, d\zeta)] ; \quad Z_0^\kappa = 1.$$ 

Let $\mathbb{K}$ denote a given set of admissible scenario controls $\kappa = (\kappa_0, \kappa_1)$, $\mathcal{F}_t$-predictable, s.t. $\kappa_1(t, z) \geq -1 + \epsilon$, and $E[\int_0^T \{ |\kappa_0^2(t) | + \int_{\mathbb{R}} \kappa_1^2(t, z) \nu(dz) \} dt] < \infty$. 

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**INNOVATION S IN STOCHASTIC ANALYSIS AND APPLICATIONS**

with emphasis on **STOCHASTIC CONTROL AND INFORMATION**

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**ERC**

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By the Girsanov theorem, using the measure $R^\kappa$ in stead of the original measure $P$ in the computations involving the price process $S(t)$, is equivalent to using the original measure $P$ in the computations involving the perturbed price process $S_\mu(t)$ in stead of $P(t)$, where $S_\mu(t)$ is given by

\[
\begin{cases}
    dS_\mu(t) = S_\mu(t^-)[(b_t + \mu_t \sigma_t)dt + \sigma_t dB_t + \int_{\mathbb{R}} \gamma(t, \zeta)\tilde{N}(dt, d\zeta)] \\
    S_\mu(0) > 0,
\end{cases}
\]

(0.38)

with

\[
\mu_t \sigma_t = -\sigma_t \kappa_0(t) - \int_{\mathbb{R}} \gamma(t, \zeta)\kappa_1(t, \zeta)\nu(d\zeta)dt
\]

(0.39)
Recall that a measure $R^\kappa$ is an Equivalent Local Martingale Measure (ELMM) iff $\kappa = (\kappa_0, \kappa_1)$ are such that

$$b_t + \sigma_t \kappa_0(t) + \int_{\mathbb{R}} \gamma_t(\zeta) \kappa_1(t, \zeta) \nu(d\zeta) = 0.$$ 

Note that we do not assume a priori that the measures $R^\kappa$ for $\kappa \in \mathbb{K}$ are ELMMs.
Robust Stochastic Control

Accordingly, we now replace the price process $S(t)$ by the perturbed process

$$
\begin{cases}
    dS_\mu(t) = S_\mu(t^-)[(b_t + \mu_t \sigma_t)dt + \sigma_t dB_t + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta)] \\
    S_\mu(0) > 0,
\end{cases}
$$

(0.40)

for some predictable perturbation process $\mu_t$, assumed to satisfy $E \left[ \int_0^T |\mu_t \sigma_t| dt \right] < \infty$. We let $M$ denote this set of processes $\mu$.

Let $X_t = X_{\varphi, \mu}(t)$ be the wealth corresponding to portfolio $\varphi$ and $\mu$, i.e.

$$
dX_t = \varphi_t S_\mu(t^-)[(b_t + \mu_t \sigma_t)dt + \sigma_t dB_t + \int_{\mathbb{R}} \gamma(t, \zeta) \tilde{N}(dt, d\zeta)]; X_0 = x > 0.
$$
Let $\mathcal{A}$ denote the set of portfolios $\varphi_t$ such that

$$E \left[ \int_0^T \varphi_t^2 S_\mu(t)^2 \left\{ (b_t + \mu_t \sigma_t)^2 + \sigma_t^2 + \int_{\mathbb{R}} \gamma^2(t, \zeta) \nu(d\zeta) \right\} dt \right] < \infty,$$

$$X_{\varphi, \mu}(t) > 0 \text{ for all } t \in [0, T], \text{ a.s.}$$

and $\exists \epsilon > 0 \text{ s.t.}$

$$E\left[ \int_0^T |X(t)|^{2+\epsilon} dt \right] < \infty$$

and

$$E[U'(X(T))^{2+\epsilon}] < \infty.$$
Let $\rho : \mathbb{R} \to \mathbb{R}$ be a convex penalty function, assumed to be $C^1$. The \textit{robust primal problem} is to find $(\hat{\phi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$ such that

$$\inf_{\mu \in \mathcal{M}} \sup_{\phi \in \mathcal{A}} I(\phi, \mu) = I(\hat{\phi}, \hat{\mu}) = \sup_{\phi \in \mathcal{A}} \inf_{\mu \in \mathcal{M}} I(\phi, \mu), \quad (0.41)$$

where

$$I(\phi, \mu) = E \left[ U(X_{\phi,\mu}(T)) + \int_0^T \rho(\mu_t) dt \right]. \quad (0.42)$$

This a stochastic differential game that we handle by using maximum principle. Define the Hamiltonian by

$$H_1(t, x, \phi, \mu, p, q, r) = \rho(\mu) + \phi S_\mu(t^) \left[ (b_t + \mu \sigma_t)p + \sigma_t q + \int_{\mathbb{R}} \gamma(t, \zeta) r(\zeta) \nu(d\zeta) \right].$$

Since $\frac{\partial H_1}{\partial x} = 0$, the BSDE for the adjoint processes $(p_1, q_1, r_1)$ is

$$dp_1(t) = q_1(t) dB_t + \int_{\mathbb{R}} r_1(t, \zeta) \tilde{N}(dt, d\zeta); \quad p_1(T) = U'(X_{\phi,\mu}(T)).$$
First order conditions for a maximum $\hat{\varphi}$ and a minimum $\hat{\mu}$:

$$(b_t + \hat{\mu}_t \sigma_t)p_1(t) + \sigma_t q_1(t) + \int_{\mathbb{R}} \gamma(t, \zeta)r_1(t, \zeta)\nu(d\zeta) = 0 \quad (0.43)$$

$$\rho'(\hat{\mu}_t) + \hat{\varphi}_t \sigma_t p_1(t) S_{\mu}(t^-) = 0 \quad (0.44)$$

Since $H_1$ is concave with respect to $\varphi$ and convex with respect to $\mu$, these first order conditions are also sufficient. Therefore we obtain the following characterization of a saddle point of (0.41):

**Theorem 2:** A pair $(\hat{\varphi}, \hat{\mu}) \in \mathcal{A} \times \mathcal{M}$ is a solution of the robust primal game problem (0.41) iff the solution $(p_1, q_1, r_1)$ of the BSDE

$$\begin{cases}
dp_1(t) = q_1(t)dB_t + \int_{\mathbb{R}} r_1(t, \zeta)\tilde{N}(dt, d\zeta) \quad ; \quad 0 \leq t \leq T \\
p_1(T) = U'(X_{\hat{\varphi},\hat{\mu}(T))}.
\end{cases}$$

satisfies (0.43), (0.44).
The robust dual problem is to find \( \hat{\theta} \in \Theta, \hat{\mu} \in \mathcal{M} \) such that

\[
\sup_{\mu \in \mathcal{M}} \sup_{\theta \in \Theta} J(\theta, \mu) = J(\hat{\theta}, \hat{\mu}) = \sup_{\theta \in \Theta} \sup_{\mu \in \mathcal{M}} J(\theta, \mu) \quad (0.46)
\]

where

\[
J(\theta, \mu) = E \left[ -V(G_\theta(T)) - \int_0^T \rho(\mu(t)) dt \right], \quad (0.47)
\]

\( V \) is the conjugate function of \( U \) and \( G(t) = G_{\theta,\mu}(t) \) is given by

\[
\begin{cases}
  dG(t) = G(t^-) \left[ \theta_0(t) dB_t + \int_{\mathbb{R}} \theta_1(t, \zeta) \tilde{N}(dt, d\zeta) \right] ; 0 \leq t \leq T \\
  G(0) = y > 0,
\end{cases} \quad (0.48)
\]

with the constraint that if \( y = 1 \), then the measure \( Q_{\theta,\mu} \) defined by

\[
dQ_{\theta,\mu} = G(T) dP \text{ on } \mathcal{F}_T
\]

is an ELMM for the perturbed price process \( S_\mu(t) \).
By the Girsanov theorem, this is equivalent to requiring that $(\theta_0, \theta_1)$ satisfies

$$b_t + \mu_t \sigma_t + \sigma_t \theta_0(t) + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(t, \zeta) \nu(d\zeta) = 0$$

(0.49)
Substituting
\[ \theta_0(t) = -\frac{1}{\sigma_t} \left[ b_t + \mu_t \sigma_t + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(t, \zeta) \nu(d, \zeta) \right] \]  \hspace{1cm} (0.50)

into (0.48) we get
\[
\begin{cases}
  dG(t) = G(t^-) \left( -\frac{1}{\sigma_t} \left[ b_t + \mu_t \sigma_t + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(t, \zeta) \nu(d, \zeta) \right] \right) dB_t \\
  \quad + \int_{\mathbb{R}} \theta_1(t, \zeta) \tilde{N}(dt, d\zeta) \quad ; \quad 0 \leq t \leq T \\
  G(0) = y > 0.
\end{cases}
\]  \hspace{1cm} (0.51)

The Hamiltonian becomes
\[
H_2(t, g, \theta_1, \mu, p, q, r) = -\rho(\mu) - \frac{gq}{\sigma_t} \left[ b_t + \mu \sigma_t + \int_{\mathbb{R}} \gamma(t, \zeta) \theta_1(\zeta) \nu(d, \zeta) \right] \\
+ g \int_{\mathbb{R}} \theta_1(\zeta) r(\zeta) \nu(d, \zeta). \]  \hspace{1cm} (0.52)
The BSDE for the adjoint processes \((p_2, q_2, r_2)\) is

\[
dp_2(t) = \left( \frac{q_2(t)}{\sigma_t} \right) \left[ b_t + \mu_t \sigma_t + \int_{\mathbb{R}} \gamma_t(\zeta) \theta_1(t, \zeta) \nu(d\zeta) \right] - \int_{\mathbb{R}} \theta_1(t, \zeta) r_2(t, \zeta) \nu(d\zeta) dt
\]
\[+ q_2(t) dB_t + \int_{\mathbb{R}} r_2(t, \zeta) \tilde{N}(dt, d\zeta) ; \quad p_2(T) = -V'(G(T)).
\]

(0.53)

The first order conditions for a maximum point \((\tilde{\theta}, \tilde{\mu})\) for \(H_2\) are

\[
(\nabla_{\theta_1} H_2 =) - \frac{q_2(t)}{\sigma_t} \gamma_t(\zeta) + r_2(t, \zeta) = 0 \]

(0.54)

\[
\left( \frac{\partial H_2}{\partial \mu} = \right) \rho'(\tilde{\mu}_t) + G_{\tilde{\theta}, \tilde{\mu}}(t) q_2(t) = 0.
\]

(0.55)
Substituting (0.54) into (0.53) we get

\[
\begin{aligned}
dp_2(t) &= \frac{q_2(t)}{\sigma_t} [b_2(t) + \tilde{\mu}_t \sigma_t] dt + q_2(t) dB_t + \int_{\mathbb{R}} r_2(t, \zeta) \tilde{N}(dt, d\zeta) ; t \in [0, T] \\
p_2(T) &= -V'(G_{\tilde{\theta}, \tilde{\mu}}(T)).
\end{aligned}
\]

(0.56)

**Theorem 4:** A pair \((\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathbb{M}\) is a solution of the robust dual problem (0.46)-(0.47) if and only the solution \((p_2, q_2, r_2)\) of the BSDE (0.56) also satisfies (0.54)-(0.55), i.e.

\[-\frac{q_2(t)}{\sigma_t} \gamma_t(\zeta) + r_2(t, \zeta) = 0\]

\[\rho'(\tilde{\mu}_t) + G_{\tilde{\theta}, \tilde{\mu}}(t) q_2(t) = 0\]
From robust primal to robust dual

We now use the characterizations above of the solutions \((\hat{\varphi}, \hat{\mu}) \in A \times \mathbb{M}\) and \((\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathbb{M}\) of the robust primal and dual problems, to find the relations between them.
Theorem: Assume $(\hat{\phi}, \hat{\mu}) \in A \times \mathbb{M}$ is a solution of the robust primal problem and let $(p_1, q_1, r_1)$ be the associated adjoint processes. Then,

$$\tilde{\mu} = \hat{\mu}$$

$$(0.57)$$

$$\tilde{\theta}_0(t) = \frac{q_1(t)}{p_1(t^{-})}; \quad \tilde{\theta}_1(t, \zeta) = \frac{r_1(t, \zeta)}{p_1(t^{-})}.$$  

$$(0.58)$$

are optimal for dual problem with initial value $y = p_1(0)$.

Moreover the optimal adjoint process $p_1$ for the robust primal problem coincides with the optimal density process $G_{\tilde{\theta}, \tilde{\mu}}$ for the robust dual problem, i.e.

$$p_1(t) = G_{\tilde{\theta}, \tilde{\mu}}(t); \quad 0 \leq t \leq T.$$  

$$(0.59)$$

In particular,

$$U'(X_{\hat{\phi}, \hat{\mu}}(T)) = G_{\tilde{\theta}, \tilde{\mu}}(T).$$  

$$(0.60)$$
From robust dual to robust primal

Theorem:
Let $(\tilde{\theta}, \tilde{\mu}) \in \Theta \times \mathcal{M}$ be optimal for the robust dual problem and let $(p_2, q_2, r_2)$ be the associated adjoint processes. Then

$$
\hat{\mu} := \tilde{\mu}
$$

$$
\hat{\phi}_t := \frac{q_2(t)}{\sigma_t S(t^-)} 1_{\sigma_t \neq 0} + \frac{r_2(t, \zeta)}{\gamma(t, \zeta) S(t^-)} 1_{\sigma_t = 0, \gamma(t, \zeta) \neq 0}; \ t \in [0, T].
$$

are optimal for primal problem with initial value $x = p_2(0)$. Moreover, the optimal adjoint process $p_2$ for the robust dual problem coincides with the optimal state process $X_{\hat{\phi}, \hat{\mu}}$ for the robust primal problem, i.e.

$$
p_2(t) = X_{\hat{\phi}, \hat{\mu}}, \ 0 \leq t \leq T \tag{0.61}
$$

In particular

$$
- V'(G_{\tilde{\theta}, \tilde{\mu}}(T)) = X_{\hat{\phi}, \hat{\mu}}(T). \tag{0.62}
$$
We study the robust primal problem

$$\inf_{\mu \in \mathcal{M}} \sup_{\varphi \in \mathcal{A}} E \left[ U(X_{\varphi,\mu}(T)) + \int_0^T \rho(\mu_t) dt \right]. \quad (0.63)$$

with no jumps. Then there is only one ELMM for the price process $S_\mu$, and the corresponding robust dual problem simplifies to

$$\sup_{\mu \in \mathcal{M}} E \left[ -V(G_\mu(T)) - \int_0^T \rho(\mu_t) dt \right], \quad \text{where} \quad (0.64)$$

$$dG_\mu(t) = -G_\mu(t^-) \left[ \frac{b_t}{\sigma_t} + \mu_t \right] dB_t; \quad G_\mu(0) = y > 0 \quad (0.65)$$
The first order conditions for the Hamiltonian reduce to:

\[ \tilde{\mu}_t = \rho'^{-1}(-G\tilde{\mu}(t)q_2(t)) \]  

(0.66)

which substituted into the adjoint BSDE gives:

\[
\begin{aligned}
dp_2(t) &= q_2(t)\left[\frac{b_t}{\sigma_t} + \rho'^{-1}(-G\tilde{\mu}(t)q_2(t))\right]dt + q_2(t)dB_t; \ t \in [0, T] \\
p_2(T) &= -V'(G\tilde{\mu}(T)).
\end{aligned}
\]

(0.67)
Now assume that

\[ U(x) = \ln x \quad \text{and} \quad \rho(\mu) = \frac{1}{2} \mu^2. \]  

(0.68)

Then \( V(y) = -\ln y - 1. \)

If \( b_t \) and \( \sigma_t \) are deterministic, we can use dynamic programming and this leads to the solution

\[ \tilde{\mu}_t = -\frac{b_t}{2\sigma_t}; \quad t \in [0, T] \]  

(0.69)

We now check that this also holds if \( b_t \) and \( \sigma_t \) are \( \mathcal{F}_t \)-adapted processes, by verifying that the system (0.65)-(0.67) is consistent:
This system is now

\[ G_{\tilde{\mu}}(t) = y \exp\left(-\int_0^t \frac{b_s}{2\sigma_s} dB_s - \frac{1}{2} \left( \frac{b_s}{2\sigma_s} \right)^2 ds \right) \] (0.70)

\[ q_2(t) = \frac{1}{G_{\tilde{\mu}}(t)} \cdot \frac{b_t}{2\sigma_t} \] (0.71)

\[ dp_2(t) = \frac{1}{G_{\tilde{\mu}}(t)} \left[ \frac{b_t}{2\sigma_t} dB_t + \left( \frac{b_t}{2\sigma_t} \right)^2 dt \right] \ ; \ p_2(T) = \frac{1}{G_{\tilde{\mu}}(T)} \] (0.72)

which gives

\[ d\left( \frac{1}{G_{\tilde{\mu}}(t)} \right) = \frac{1}{G_{\tilde{\mu}}(t)} \left[ \frac{b_t}{2\sigma_t} dB_t + \left( \frac{b_t}{2\sigma_t} \right)^2 dt \right] \] (0.73)
We see that (0.72) is in agreement with (0.73), so

\[ \tilde{\mu}_t = -\frac{b_t}{2\sigma_t} \]

is indeed optimal. The corresponding optimal portfolio for the robust primal is

\[ \hat{\phi}_t = \frac{b_t}{G_{\tilde{\mu}}(t)2\sigma_t S_{\tilde{\mu}}(t^{-})}; \quad t \in [0, T]. \]  

(0.74)
We have proved:

**Theorem**

Suppose (0.68) hold. Then the optimal scenario $\hat{\mu} = \tilde{\mu}$ and optimal $\hat{\phi}$ for the robust primal problem (0.63) are given by

$$\tilde{\mu}_t = -\frac{b_t}{2\sigma_t}; \quad t \in [0, T] \quad (0.75)$$

and

$$\hat{\phi}_t = \frac{b_t}{G_{\tilde{\mu}}(t)2\sigma_t S_{\tilde{\mu}}(t^-)}; \quad t \in [0, T]. \quad (0.76)$$

respectively, with $G_{\tilde{\mu}}(t)$ as in (0.70).

It is interesting to compare this result with the solution we found in the corresponding previous example in the non-robust case ($\rho = 0$):

$$\hat{\phi}(t) = X_{\hat{\phi}}(t)\frac{b_t}{\sigma_t^2 S(t^-)} \quad (0.77)$$
Summary

- **The purpose** of this presentation has been to use stochastic control theory to obtain new results and new proofs of results in portfolio optimization both with and without model uncertainty. In the case with no model uncertainty, then some of the results have been proved earlier by using convex duality theory.

- **The advantage** of this approach is that it gives an explicit relation between the optimal measure in the dual formulation and the optimal portfolio in the primal formulation, both in the cases with and without model uncertainty. BSDEs play a crucial role in this connection.


