Measuring and analysing marginal systemic risk contribution using copula

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Background

- Financial system
- Financial regulation
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- Value-at-Risk (as special case of regulatory risk measure)
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Financial system

Financial Intermediaries:
- Credit Institutions
- Other monetary institutions

Lender-Savers:
- Households
- Firms
- Central Banks

Financial Markets:
- Money and Capital Markets
- Debt and Equity Markets
- Over-the-Counter Markets

Borrower-Spenders:
- Firms
- Government
- Households

Direct Finance
Indirect Finance
The **aim** of financial regulation is to ensure the **stability of the entire financial system** (and not that of **financial institutions in isolation**).

The principle of the financial regulation before the last crisis is to impose each bank to hold (or invested in risk-free assets) a given amount (**risk capital**) in order to avoid its failure (**to reduce the probability of failure**).
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- The risk capital of a given financial institution is computing depending on its loss $L$ using **risk measures** (e.g. Value-at-Risk in Basell II).
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Value-at-Risk

**Definition: Value-at-Risk [McNeil and Frey and Embrechts(2005)]**

Given some confidence level $\alpha \in (0, 1)$. The $VaR$ of a portfolio at the confidence level $\alpha$ is given by the smallest number $l$ such that the probability that the loss $L$ exceeds $l$ is no larger than $(1 - \alpha)$. Formally

$$VaR_\alpha := \inf \{ l \in \mathbb{R} : Pr(L > l) \leq 1 - \alpha \}$$

$$= \inf \{ l \in \mathbb{R} : Pr(L \leq l) \geq \alpha \}.$$

Recall that, for a distribution function $F$, the function defined by

$$q_\alpha (F) := \inf \{ x \in \mathbb{R} : F(x) \geq \alpha \}.$$

is called the $\alpha$-quantile of $F$.

Note that, if $F$ is continuous and strictly increasing, we have

$$q_\alpha (F) = F^{-1} (\alpha).$$
We consider here only random variables which have strictly positive density function, such that all distribution function $F$ considered here are continuous and strictly increasing.

We have in this case

$$\text{VaR}_\alpha = F^{-1}(\alpha).$$ (1)
Measuring systemic risk contribution

- Systemic risk
- Regulation of systemic risk
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- CoVaR-Method
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Lessons from the Financial Crisis

With the last crisis it became clear that the failure of certain financial institutions (the so called system relevant financial institutions) can lead to failures by other financial institutions threatening in this way the stability of the financial system (systemic risk).

A typical example is the failure of Lehman Brothers in Sept. 15th. 2008.

Figure: Bank Failures in the United States, from 2000 to 2010 (Source: FIDC)
Which factors promote systemic risk?

1) The high interconnectedness and complexity of the modern financial system
2) The size of financial institutions
3) The non regulation of systemic risk (no systemic risk measure)
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3) The non regulation of systemic risk (no systemic risk measure)

Remark
1), 2) and 3) ⇒ Too big to fail-, too interconnected to fail-theory. According to this, certain financial institutions are so large and so interconnected that their failure will be disastrous to the whole financial system (Moral hazard, contagion)
2) The size of financial institutions
3) The non regulation of systemic risk (no systemic risk measure)

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Regulation of systemic risk

The main question here is:

- How to quantify the systemic risk contribution of one single financial institute to the whole financial system.
- or how to quantify the adverse financial impact that the failure of one given financial institution can cause to the whole financial system.

Probabilistic framework

- We denote by $i$ and $s$ the individual financial institution and the financial system respectively.
- We assume that the losses of $i$ and $s$ are modeled by the r.v. $L^i$ and $L^s$ respectively.
- We assume that $i$ and $s$ are interconnected and thus non-independent such the bivariate r.v. $(L^i, L^s)$ is assumed to be statistically dependent.
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As response to this question [Adrian and Brunnermeier(2011)] proposed the so called \textit{CoVaR-method}.

The \textit{CoVaR-method} build on the term $CoVaR_{\alpha}^{s|C(L^i)}$.

\textbf{Definition}

$CoVaR_{\alpha}^{s|C(L^i)}$ is defined as the Value-at-Risk at the level $\alpha$ of an financial institution $s$ (or a financial system) conditional on some event $C(L^i)$ depending on the loss $L^i$.

$$Pr \left( L^s \leq CoVaR_{\alpha}^{s|C(L^i)} | C(L^i) \right) = \alpha. \quad (2)$$
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$\text{CoVaR}_{\alpha}^{s|C(L^i)}$ is thus as a **conditional quantile** of the loss distribution of the system $s$. 
In the CoVaR-method the systemic risk contribution of a single institution $i$ is modeled (estimated) by the risk measure $\Delta\text{CoVaR}_\alpha^{s|i}$.

**Definition**

$$\Delta\text{CoVaR}_\alpha^{s|i} := \text{CoVaR}_\alpha^{s|L^i=\text{VaR}^i_\alpha} - \text{CoVaR}_\alpha^{s|L^i=E(L^i)}.$$ 

So, the main task here is the computation of the generalized $\text{CoVaR}_\alpha^{s|L^i=l}$, $l \in \mathbb{R}$ i.e.

$$\text{Pr} \left( L^s \leq \text{CoVaR}_\alpha^{s|L^i=l} | L^i = l \right) = \alpha. \quad (3)$$

Recall that $\text{Pr} \left( L^i = l \right) = 0$, for any $l \in \mathbb{R}$. However for fixed a $h$ a conditional probability of the form $\text{Pr} \left( L^s \leq h | L^i = l \right)$ can be defined as

$$\forall y \in \mathbb{R}, \quad \text{Pr} \left( L^s \leq h | L^i \leq y \right) = \int_{-\infty}^{y} \text{Pr} \left( L^s \leq h | L^i = l \right) f_i(l) \, dl \quad (4)$$
Computation of $CoVaR^s_{\alpha|L^i = l}$

- Method based on quantile regression.
- Closed form formula under bivariate Gaussian distribution setting.
[Adrian and Brunnermeier (2011)] adopt a quantile regression approach as described in Koenker (1978) “Regression Quantiles”

1. Statical approach (i.e. no closed form or analytical formula)
2. Impose the bivariate normal distribution as a model for the system and the firm variables.
Quantile regression approach

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Closed form formula when \((L^i, L^s)\) is modeled by a bivariate Gaussian r.v.

[Jäger-Ambrożewicz(2010)] proposed a closed form formula for \(\text{CoVaR}^s_{\alpha} \mid L^i = l\) by assuming that \((L^i, L^s)\) follows a bivariate Gaussian distribution

\[
\begin{pmatrix} L^s \\ L^i \end{pmatrix} \sim \mathcal{N}_2(\mu, \Sigma). \quad \text{with} \quad \mu = \begin{pmatrix} \mu_s \\ \mu_i \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma^2_s & \rho \sigma_s \sigma_i \\ \rho \sigma_s \sigma_i & \sigma^2_i \end{pmatrix}.
\]

Where \(\rho\) is the correlation between \(L^i\) and \(L^s\). In this case, it is well known that the conditional distributions of \(L^s\) given \(L^i\) assume a value \(l\) is univariate Gaussian distributed

\[
L^s \mid L^i = l \sim \mathcal{N} \left( \mu_s + \rho \frac{\sigma_s}{\sigma_i} (l^i - \mu_s), \sigma^2_s \left( 1 - \rho^2 \right) \right)
\]

(cf. e.g. [McNeil and Frey and Embrechts(2005)]).

Hence \(\text{CoVaR}^s_{\alpha} \mid L^i = l\) is

\[
\text{CoVaR}^s_{\alpha} \mid L^i = l = \sigma_s \sqrt{1 - \rho^2} \Phi^{-1} (\alpha) + \mu_s + \rho \frac{\sigma_s}{\sigma_i} (l^i - \mu_s).
\]

Where \(\Phi\) is the distribution function of the standard Gaussian distribution.
Our approach: Computing $\text{CoVaR}_\alpha^{s|L^i=L}$ through Copula

- Motivation
- Copula
- Our principal result (our formula)
The computation method presented above have their relative advantages and disadvantages but they share the common restriction that both impose bivariate Gaussian distribution as model for \((L^i, L^s)\). There is a major reasons why bivariate Gaussian distribution can lead to difficulties.

So, we assert that the \(\text{CoVaR}^s_{\alpha | L^i = l} \) computed under the assumption that \((L^i, L^s)\) is bivariate Gaussian distributed is not consistent with the notion of systemic risk.
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- Recall that the aim of systemic measurement is to quantify the risk contribution of **distressed financial institution**
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- Recall that the aim of systemic measurement is to quantify the risk contribution of distressed financial institution
- In general institution defaults and systemic crisis can be considered as extreme event. Indeed, the default which produces the contagion effect corresponds generally to a shock (large loss) relative to an expected loss. This can be characterized by an extreme value which appears in the tail of the corresponding loss distributions.
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- Recall that the aim of systemic measurement is to quantify the risk contribution of **distressed financial institution**.
- In general institution defaults and systemic crisis can be considered as extreme event. Indeed, the default which produces the contagion effect corresponds generally to a shock (large loss) relative to an expected loss. This can be characterized by **an extreme value** which appears **in the tail** of the corresponding loss distributions.
Hence, in the context of the analysis and the measurement of the systemic risk contribution of the financial institution $i$, the distributions of $L^i$ and $L^s$ have to be dependent in their tail. The tail dependence of $i$ and $s$ have to be considered. This can be verified by a tail dependence measure e.g. the tail dependence coefficient $\lambda$. 
Tail dependence coefficient

**Definition [cf. [McNeil and Frey and Embrechts(2005)]]**

Let \((X, Y)\) be a bivariate random variable with marginal distribution functions \(F\) and \(G\), respectively. The upper tail dependence coefficient of \(X\) and \(Y\) is the limit (if it exists) of the conditional probability that \(Y\) is greater than the \(100\alpha - th\) percentile of \(G\) given that \(X\) is greater than the \(100\alpha - th\) percentile of \(F\) as \(\alpha\) approaches 1, i.e.

\[
\lambda_u := \lim_{\alpha \to 1^-} Pr\left( Y > G^{-1}(\alpha) \mid X > F^{-1}(\alpha) \right). \tag{5}
\]

\(\lambda_u\) measures the probability that \(Y\) exceeds the threshold \(G^{-1}(\alpha)\), conditional on that \(X\) exceeds the threshold \(F^{-1}(\alpha)\). In other words, \(\lambda_u\) measures the tendency for extreme events to occur simultaneously and thus in some sense also the contagion effect.

Hence, if \((L^i, L^s)\) is modeled such that \(L^i\) and \(L^s\) are asymptotically independent in the upper tail \((\lambda_u = 0)\) then extreme losses appear to occur independently and they is in this case **no contagion effect**.
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Examples

- If \((L^i, L^s)\) is modeled by bivariate Gaussian r.v. then
  \[
  \lambda_u = \begin{cases} 
  0 & \text{if } \rho < 1 \\
  1 & \text{if } \rho = 1.
  \end{cases}
  \]

  This means that the bivariate Gaussian distribution is **unappropriated (unrealistic)** for the analysis of systemic risk.

- If \((L^i, L^s)\) is modeled by bivariate t-student r.v. then
  \[
  \lambda_u = \begin{cases} 
  > 0 & \text{if } \rho > -1 \\
  0 & \text{if } \rho = -1
  \end{cases}
  \]

  The bivariate \(t\)-student distribution is thus a better alternative model for systemic risk analysis.
A 2-dimensional copula is a (distribution) function $C : [0, 1]^2 \rightarrow [0, 1]$ with the following satisfying:

- **Boundary conditions:**
  1) For every $u \in [0, 1] : C(0, u) = C(u, 0) = 0$.
  2) For every $u \in [0, 1] : C(1, u) = u$ and $C(u, 1) = u$.

- **Monotonicity condition:**
  3) For every $(u_1, u_2), (v_1, v_2) \in [0, 1] \times [0, 1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$
  
  $$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$ 

Conditions 1) and 3) implies that the so defined 2-copula $C$ is a bivariate joint distribution function (cf. [Nelsen(2006)]) and Condition 2) implies that the copula $C$ has standard uniform margins.
Theorem [Nelsen(2006)] (Continuity)

For every \( u_1, u_2, v_1, v_2 \in [0, 1] \) with \( u_1 < u_2 \) and \( v_1 < v_2 \)

\[
|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|
\] (6)

This means that copulas are lipschitz continuous with Lipschitz constant equal to 1.

Theorem [Nelsen(2006)] (Differentiability)

Let \( C \) be a copula. For any \( v \in [0, 1] \), the partial derivative \( \partial C(u, v)/\partial u \) exists for almost all \( u \), and for such \( v \) and \( u \)

\[
0 \leq \frac{\partial C(u, v)}{\partial u} \leq 1.
\]

Similarly, for any \( u \in [0, 1] \), the partial derivative \( \partial C(u, v)/\partial v \) exists for almost all \( v \), and for such \( u \) and \( v \)

\[
0 \leq \frac{\partial C(u, v)}{\partial v} \leq 1.
\]
Sklar’s theorem

Let $H$ be a joint distribution function with marginal distribution functions $F$ and $G$, then there exists a copula $C$ such that for all $x, y \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

$$H(x, y) = C[F(x), G(y)]. \quad (7)$$

If $F$ and $G$ have density, then $C$ is unique. Conversely, if $C$ is a copula and $F$ and $G$ are distribution functions, then the function $H$ defined by (7) is a joint distribution function with margins $F$ and $G$.

Corollary [Nelsen(2006)]

Let $H$ denote a bivariate distribution function with margins $F$ and $G$ satisfying our assumption, then there exist a unique copula $C$ such that for all $(u, v) \in [0, 1]^2$ it holds:

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)).$$
Interpretation of the Sklar’s theorem

1. The Copula - method allows the effectively separation of the dependencies from the margins

\[ F_{X,Y}(x,y) \]

\[ C(u,v) \]

\[ F_X(x), F_Y(y) \]

2. The Copula can be interpreted as the information missing from the individual margins to complete the joint distribution.

\[ C_{X,Y}(u,v) \]

\[ F_X(x), F_Y(y) \]

\[ F_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y)) \]
Copula's Type

Depending on the way they are build. Copulas can be classify into two classes

1. Implicit copulas. i.e copulas which are derived from a multivariate distribution . (e.g. elliptical copulas). Implicit bivariate copulas have in general the form

\[
C(u, v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f(s, t) \, ds \, dt,
\]

where \( f(s, t) \) is the density of the corresponding bivariate distribution.

2. Explicit copulas. i.e. copulas which are not derived from a multivariate distribution. (e.g. Archimedean copulas)

Remark

Implicit copula with corresponding margins corresponds to the underlying multivariate distribution.
The bivariate Gaussian copula is defined as follows [Nelsen(2006)]

\[
C_{\rho}(u, v) = \Phi_2 \left( \Phi(u)^{-1}, \Phi(v)^{-1} \right) \\
= \int_{-\infty}^{u} \int_{-\infty}^{v} \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left( \frac{2\rho st - s^2 - t^2}{2(1 - \rho^2)} \right) dsdt
\]

where \(\Phi_2\) is the bivariate standard normal distribution with linear correlation coefficient \(\rho\), and \(\Phi\) the univariate standard normal distribution.

The bivariate t copula with \(\nu\) degrees of freedom is defined as

\[
C_{t,\nu}(u, v) = t_{\rho,\nu} \left( t_{\nu}^{-1}(u), t_{\nu}^{-1}(v) \right) \\
= \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{1}{2\pi \sqrt{1 - \rho^2}} \left( 1 + \frac{s^2 + t^2 - 2\rho st}{\nu(1 - \rho^2)} \right)^{-\frac{\nu + 2}{2}} dsdt.
\]

t_{\nu}(x) is the univariate t-distribution with \(\nu\) degrees of freedom.

These two copulas have in the central part the same behavior but show different behaviors in the tail. This difference vanish with increasing \(\nu\).
**Implicit copulas: examples of elliptical copulas**

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\[ = \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{1}{2\pi \sqrt{1 - \rho^2}} \left( 1 + \frac{s^2 + t^2 - 2\rho st}{\nu (1 - \rho^2)} \right)^{-\frac{\nu + 2}{2}} ds dt. \]

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These two copulas have in the central part the same behavior but show different behaviors in the tail. This difference vanish with increasing \( \nu \).
Explicit copulas: Archimedean copulas

Definition [McNeil and Frey and Embrechts (2005)]

A copula of the form

\[ C(u, v) = \varphi^{-1} (\varphi(u) + \varphi(v)) \, . \] (8)

is an Archimedean copula if \( \varphi \) is a convex and strictly decreasing continuous function from \([0, 1]\) to \([0, \infty]\) with \( \varphi(1) = 0 \). \( \varphi \) is called the generator of the corresponding Archimedean copula.

If \( \varphi(0) = \infty \) the generator is said to be strict and it is equivalent to the ordinary functional inverse \( \varphi^{-1} \).

So, to construct a Archimedean copulas we need only to find a generator function and than define the corresponding copulas through Equation (8).
Our principal result

Theorem [Hakwa and Jäger-Ambrożewicz and Rüdiger(2011)]
(Generalized explicit formula for $\text{CoVaR}_\alpha^{s|L^i=l}$)

Let $L^i$ and $L^s$ be two random variables representing the loss of the financial institution $i$ and that of the financial system $s$ respectively. Assume that the joint distribution of $(L^i, L^s)$ is defined by a bivariate copula $C$ with marginal distribution functions $F_i$ and $F_s$ respectively.

If $F_i$ and $F_s$ are continuous, strictly increasing and strictly positive and the partial derivative

$$g(v, u) := \frac{\partial C(u, v)}{\partial u}$$

is invertible with respect to the parameter $v$, then for all $l \in \mathbb{R}$ the explicit formula for $\text{CoVaR}_\alpha^{s|L^i=l}$ at level $\alpha$, $0 < \alpha < 1$ is given by

$$\text{CoVaR}_\alpha^{s|L^i=l} = F_s^{-1} \left( g^{-1} \left( \alpha, F_i (l) \right) \right).$$

(9)
Remark

Like a Copula, $\text{CoVaR}_\alpha^{s|L^i=l}$ can be separated into two distinct components.

1. The margins $L^s$ and $L^i$, which represent the purely univariate features of the system $s$ and the single Firm $i$ respectively.

2. The function $g^{-1}$, which represents the true interconnection between $i$ and $s$. 
Assume that the copula of $L^i$ and $L^s$ is the Gaussian copula, then

$$\text{CoVaR}^s_{\alpha|L^i=l} = F_s^{-1} \left( \Phi \left( \rho \Phi^{-1} (F_i(l)) + \sqrt{1 - \rho^2} \Phi^{-1} (\alpha) \right) \right).$$

where $F_i$ and $F_s$ represent the univariate distribution function of $L^i$ and $L^s$ respectively.

In particular if $L^i$ and $L^s$ are univariat Gaussian distributed then

$$\text{CoVaR}^s_{\alpha|L^i=l} = \rho \frac{\sigma_s}{\sigma_i} (l - \mu_i) + \sqrt{1 - \rho^2} \sigma_s \Phi^{-1} (\alpha) + \mu_s.$$

This show that that the formula proposed by [Jäger-Ambrożewicz(2010)] is a special case of our generalized formula.
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In particular if $L^i$ and $L^s$ are univariate Gaussian distributed then

$$\text{CoVaR}_{\alpha}^{s|L^i=l} = \rho \frac{\sigma_s}{\sigma_i} (l - \mu_i) + \sqrt{1 - \rho^2} \sigma_s \Phi^{-1} (\alpha) + \mu_s.$$ 

This show that the formula proposed by [Jäger-Ambrożewicz(2010)] is a special case of our generalized formula.
Example: Bivariate t copula

If we assume the bivariate student t Copula as modeled for \((L^i, L^s)\) then

\[
CoVaR^s_{\alpha | L^i = l} = F^{-1}_s \left( t_{\nu} \left( \rho t^{-1}_\nu (F_i(l)) + \sqrt{\frac{(1 - \rho^2)(\nu + [t^{-1}_\nu (F_i(l))]^2)}{\nu + 1}} \right) \right)
\]

In particular if \(L^i\) and \(L^s\) are univariate t distributed with degrees of freedom \(\nu\) then

\[
CoVaR^s_{\alpha | L^i = l} = (\rho \cdot l) + \sqrt{\frac{(1 - \rho^2)(\nu + l^2)}{\nu + 1}} t^{-1}_{\nu+1}(\alpha)
\]
Example: Bivariate t copula

If we assume the bivariate student t Copula as modeled for \((L^i, L^s)\) then

\[
CoVaR_{\alpha}^{s|L^i=l} = F_s^{-1}\left( t_\nu \left( \rho t_{\nu}^{-1} (F_i(l)) + \sqrt{\frac{(1 - \rho^2)(\nu + [t_{\nu}^{-1} (F_i(l))]^2)}{\nu + 1}} t_{\nu+1}^{-1}(\alpha) \right) \right)
\]

In particular if \(L^i\) and \(L^s\) are univariate t distributed with degrees of freedom \(\nu\) then

\[
CoVaR_{\alpha}^{s|L^i=l} = (\rho \cdot l) + \sqrt{\frac{(1 - \rho^2)(\nu + l^2)}{\nu + 1}} t_{\nu+1}^{-1}(\alpha)
\]
Proposition: [Hakwa and Jäger-Ambrożewicz and Rüdiger(2011)]

$CoVaR^s_{\alpha | L^i = l}$ for Archimedean copula

Let $L^i$ and $L^s$ be two random variables representing the loss of the financial institution $i$ and that of the financial system $s$. Assume that the joint distribution of $(L^i, L^s)$ is defined by a bivariate copula $C$ with marginal distribution functions $F_i$ and $F_s$ respectively.

If $C$ is an Archimedean copula with a continuous, strictly, decreasing and convex generator $\varphi$, then the explicit formula for the $CoVaR^s_{\alpha | L^i = l}$ at level $\alpha$, $0 < \alpha < 1$

$$CoVaR^s_{\alpha | L^i = l} = F_s^{-1} \left( g^{-1} (\alpha, F_i (l)) \right).$$

$$= F_s^{-1} \left( \varphi^{-1} \left( \varphi \left( \varphi^{-1} \left( \frac{\varphi' (F_i (l))}{\alpha} \right) \right) - \varphi (F_i (l)) \right) \right)$$

provided that the function

$$g (v, u) := \frac{\partial C (u, v)}{\partial u}$$
**Proof of the theorem**

Recall that the implicit definition of $\text{CoVaR}_{\alpha}^{s|L^i=l}$ is given by:

\[
\Pr\left( L^s \leq \text{CoVaR}_{\alpha}^{s|L^i=l} | L^i = l \right) = \alpha
\]

\[
\iff \Pr\left( F_s(L^s) \leq F_s\left( \text{CoVaR}_{\alpha}^{s|L^i=l} \right) | F_i(L^i) = F_i(I) \right) = \alpha.
\]

Let $V = F_s(L^s)$, $U = F_i(L^i)$, $v = F_s\left( \text{CoVaR}_{\alpha}^{s|L^i=l} \right)$ and $u = F_i(I)$

i.e.

\[
\Pr\left( L^s \leq \text{CoVaR}_{\alpha}^{s|L^i=l} | L^i = l \right) = \Pr\left( F_s(L^s) \leq F_s\left( \text{CoVaR}_{\alpha}^{s|L^i=l} \right) | F_i(L^i) = F_i(I) \right)
\]

\[
= \Pr\left( V \leq v | U = u \right).
\]

Due to Assumption 1 it follows from Remark ?? that $V$ and $U$ are standard uniform distributed. In this case we can refer to ([Breiman(1992)]) and ([Roncalli(2009)]) and compute the conditional
probability \( Pr( V \leq v \mid U = u ) \), as follows:

\[
Pr( V \leq v \mid U = u ) = \lim_{\Delta u \to 0^+} \frac{Pr( V \leq v, u \leq U \leq u + \Delta u )}{Pr( u \leq U \leq u + \Delta u )}
\]

\[
= \lim_{\Delta u \to 0^+} \frac{Pr( U \leq u + \Delta u, V \leq v ) - Pr( U \leq u, V \leq v )}{Pr( U \leq u + \Delta u ) - Pr( U \leq u )}
\]

\[
= \lim_{\Delta u \to 0^+} \frac{C( u + \Delta u, v ) - C( u, v )}{\Delta u}
\]

\[
= \frac{\partial C( u, v )}{\partial u} =: g( u, v ).
\]

assume that \( g \) is invertible with respect to the (non-conditioning) variable \( v \), then

\[
v = g^{-1}( \alpha, u ).
\]

Set \( v = F_s \left( CoVaR_{\alpha}^{s|L^i=l} \right) \) and \( u = F_i(l) \). We obtain

\[
F_s \left( CoVaR_{\alpha}^{s|L^i=l} \right) = g^{-1}( \alpha, F_i(l) ).
\]

Thus

\[
CoVaR_{\alpha}^{s|L^i=l} = F_s^{-1} \left( g^{-1}( \alpha, F_i(l) ) \right).
\]
Using copula theories we provide a general closed formula for the computation of \( \text{CoVaR}_\alpha^{s|L^i=l} \).

Our formula coincide with the formula proposed by [Jäger-Ambrożewicz(2010)] when \((L^i, L^s)\) is modeled by a Bivariate Gaussian copula with Gaussian margins. (i.e. it is a special case of our formula)

Our formula generates a characterization of \( \text{CoVaR}_\alpha^{s|L^i=l} \) in term of a generator when the dependence between \( L^i \) and \( L^s \) is modeled by a Archimedean copula.

Our formula provides a framework for flexible modeling and analysis of systemic risk contribution by allowing the integration of stylized features of marginal losses as skewness, fat-tails and complex dependence structure (e.g. linear-, non-linear-, tail-dependence)
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