Quantum White Noise Derivatives and Their Applications

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Hammamet, October 15, 2013
Plan

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1. Elements of Quantum White Noise Calculus
1.1. Boson Fock Space

\( T \): a topological space (time interval, space-time manifold, even a discrete space,...)

\( H = L^2(T, dt) \): Hilbert space of \( \mathbb{C} \)-valued \( L^2 \)-functions

The Boson Fock space over \( H = L^2(T) \) is defined by

\[
\Gamma(H) = \left\{ \phi = (f_n); f_n \in H^\bigotimes n, \ ||\phi||^2 = \sum_{n=0}^{\infty} n!|f_n|^2_0 < \infty \right\},
\]

where \( |f_n|_0 \) is the usual \( L^2 \)-norm of \( H^\bigotimes n = L^2_{\text{sym}}(T^n) \).

Annihilation and Creation Operators

\[
A(f) : (0, \ldots, 0, \xi^\bigotimes n, 0, \ldots) \mapsto (0, \ldots, 0, n \langle f, \xi \rangle \xi^\bigotimes (n-1), 0, 0, \ldots)
\]

\[
A^*(f) : (0, \ldots, 0, \xi^\bigotimes n, 0, \ldots) \mapsto (0, \ldots, 0, 0, \xi^\bigotimes n \hat{\otimes} f, 0, \ldots)
\]

\( \triangleright \) \( A(f) \) and \( A(f)^* \) are unbounded operators in \( \Gamma(H) \).

\[
A(f) = \int_T f(t) a_t \, dt, \quad A^*(f) = \int_T f(t) a_t^* \, dt,
\]

are merely symbolic notation \( \implies \) Formulate \( a_t \) and \( a_t^* \) as continuous operators
1.2. White Noise Distributions

Gelfand (nuclear) triple for $H = L^2(T)$

$$E \subset H = L^2(T) \subset E^*,$$

$$E = \operatorname{proj lim}_{p \to \infty} E_p, \quad E^* = \operatorname{ind lim}_{p \to \infty} E_{-p},$$

where $E_p$ is a dense subspace of $H$ and is a Hilbert space for itself.

Gelfand (nuclear) triple for $\Gamma(H)$ [Kubo–Takenaka PJA 56A (1980)]

$$(E) \subset \Gamma(H) \subset (E)^*, \quad (E) = \operatorname{proj lim}_{p \to \infty} \Gamma(E_p), \quad (E)^* = \operatorname{ind lim}_{p \to \infty} \Gamma(E_{-p})$$

The annihilation and creation operator at a point $t \in T$

$$a_t : (0, \ldots, 0, \xi \otimes^n, 0, \ldots) \mapsto (0, \ldots, 0, n\xi(t)\xi \otimes^{(n-1)}, 0, 0, \ldots)$$

$$a^*_t : (0, \ldots, 0, \xi \otimes^n, 0, \ldots) \mapsto (0, \ldots, 0, 0, \xi \otimes^n \delta_t, 0, \ldots)$$

The pair $\{a_t, a^*_t ; t \in T\}$ is called the quantum white noise on $T$. $a_t \in \mathcal{L}((E), (E))$ and $a^*_t \in \mathcal{L}((E)^*, (E)^*)$ for all $t \in \mathbb{R}$. Moreover, both maps $t \mapsto a_t \in \mathcal{L}((E), (E))$ and $t \mapsto a^*_t \in \mathcal{L}((E)^*, (E)^*)$ are operator-valued test functions, i.e., belongs to $E \otimes \mathcal{L}((E), (E))$ and $E \otimes \mathcal{L}((E)^*, (E)^*)$, respectively.
1.3. Classical and Quantum Brownian Motions

\[ E = S(\mathbb{R}) = \operatorname{proj lim}_{p \to \infty} S_p(\mathbb{R}) \text{ and } H = L^2(\mathbb{R}). \]

\( \mu \): Gaussian measure on \( E^* \) defined by the characteristic functional:

\[ e^{-\|\xi\|^2/2} = \int_{E^*} e^{i\langle x, \xi \rangle} \mu(dx), \quad \xi \in E. \]

\((E^*, \mu)\): Gaussian space

Under the Wiener–Itô–Segal isomorphism \( L^2(E^*, \mu) \cong \Gamma_{\text{Boson}}(H) \) the Brownian motion is defined by

\[ B_t = \langle x, 1_{[0,t]} \rangle \leftrightarrow (0, 1_{[0,t]}, 0, 0, \ldots) \]

Quantum decomposition of Brownian motion and white noise:

\[ B_t = A(1_{[0,t]}) + A^*(1_{[0,t]}) \]

\[ W_t = a_t + a_t^* \]

\( t \mapsto W_t \in (E)^* \) white noise process (Hida’s idea)

\( t \mapsto a_t \in \mathcal{L}((E), (E)), a_t^* \in \mathcal{L}((E)^*, (E)^*) \) quantum white noise process
1.4. CKS- and GHOR-Approaches for $\mathcal{W} \subset \Gamma(H) \cong L^2(E^*, \mu) \subset \mathcal{W}^*$


$$\Gamma_{\alpha}(E_p) = \left\{ \phi = (f_n); f_n \in E_p^{\hat{\otimes}n}, \|\phi\|_{p,+}^2 = \sum_{n=0}^{\infty} n! \alpha(n) |f_n|^2_p < \infty \right\},$$

$$\mathcal{W} = \operatorname{proj \lim}_{p \to \infty} \Gamma_{\alpha}(E_p),$$


$$\operatorname{Exp}(E_p, \theta, m) = \left\{ f : E_p \to \mathbb{C}; \text{ entire holomorphic,} \right. $$

$$\|f\|_{\theta, p, m} = \sup_{x \in E_p} |f(x)| e^{-\theta(m|x|_p)} < \infty$$

$$\mathcal{F}_\theta(E^*) = \operatorname{proj \lim}_{p \to \infty, m \to +0} \operatorname{Exp}(E_{-p}, \theta, m),$$

$$\mathcal{W} = \{ \text{Taylor coefficients of } \phi \in \mathcal{F}_\theta(E^*) \}$$


The above two classes of Gelfand triples coincide.
1.5. White Noise Operators

**Definition**

Based on the Gelfand triple $(E) \subset \Gamma(H) \subset (E)^*$ a continuous operator from $(E)$ into $(E)^*$ is called a *white noise operator*. The space of white noise operators is denoted by $\mathcal{L}((E), (E)^*)$ (bounded convergence topology).

$\mathcal{L}((E), (E)), \mathcal{L}((E)^*, (E)^*), \mathcal{L}(\Gamma(H), \Gamma(H)) \subset \mathcal{L}((E), (E)^*)$.

**Definition (Integral kernel operator)**

Given $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$, $l, m = 0, 1, 2, \ldots$,

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \ldots, s_l, t_1, \ldots, t_m)$$

$$a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} \, ds_1 \cdots ds_l \, dt_1 \cdots dt_m$$

is a well-defined white noise operator and is called an *integral kernel operator*.

$$\Delta_G = \int_T a_t^2 \, dt \quad \text{(Gross Laplacian)} \quad N = \int_T a_t^* a_t \, dt \quad \text{(Number operator)}$$
1.6. Fock Expansion

**Theorem (O. JMSJ 45 (1993); tracing back to Haag, Berezin, Krée,...)**

*Every white noise operator* \( \Xi \in \mathcal{L}((E), (E)^*) \) *admits the Fock expansion:*

\[
\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \quad \kappa_{l,m} \in (E^{\otimes (l+m)})^*,
\]

*where the right-hand side converges in* \( \mathcal{L}((E), (E)^*) \). *If* \( \Xi \in \mathcal{L}((E), (E)) \), *then* \( \kappa_{l,m} \in E^{\otimes l} \otimes (E^{\otimes m})^* \) *and the series converges in* \( \mathcal{L}((E), (E)) \).

**Applications and Generalizations:**

1. **rotation-invariant operators** \((N, \Delta_G, \Delta_G^*) \) \([O. MZ 210 (1992)]\)
2. **derivations and Wick derivations** \([Chung–Chung (1996), Huang–Luo (1998), etc.]\)
3. **multilinear operators** \([Ji–O.–Ouerdiane (2002)]\)
4. **Hochschild cohomology group** \([Léandre (2008)]\)

**Our Conceptual Standpoint**

A white noise operator \( \Xi \) as a function of quantum white noise:

\[
\Xi = \Xi(a_s, a_t^*; s, t \in T)
\]

\(\implies\) *We treat* \( \{a_s, a_t^*; s, t \in T\} \) *as a coordinate system* for white noise operators.
1.7. Wick Product

Let us introduce a product of operators, different from the usual composition.

**Definition (Wick (normal-ordered) product)**

For $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$ the Wick (or normal-ordered) product $\Xi_1 \diamond \Xi_2$ is defined by

$$(\Xi_1 \diamond \Xi_2)(\xi, \eta) = \tilde{\Xi}_1(\xi, \eta)\tilde{\Xi}_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in E,$$

where $\tilde{\Xi}(\xi, \eta)$ is the symbol of a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ defined by

$$\tilde{\Xi}(\xi, \eta) = \langle \Xi \phi_{\xi}, \phi_{\eta} \rangle, \quad \xi, \eta \in E,$$

where $\phi_{\xi} = (1, \xi, \cdots, \xi \otimes n / n!, \cdots)$ is an exponential vector.

Important properties:

1. Equipped with the Wick product, $\mathcal{L}((E), (E)^*)$ becomes a commutative algebra.
2. For any $\Xi \in \mathcal{L}((E), (E)^*)$ we have

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t, \quad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^* \Xi.$$
1.8. Convolution Product $\equiv$ Wick Product

**Definition (Ben Chrouda–El Oued–Ouerdiane Soochow JM 28 (2002))**

With each $\Phi \in \mathcal{W}^*$ we associate the *convolution operator* $C_\Phi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ defined by

$$[H(C_\Phi \phi)](x) = \langle \langle \Phi, T_{-x} \phi \rangle \rangle, \quad x \in E^*.$$ 

**Theorem (O.–Ouerdiane IDAQP 14 (2011))**

$$C_\Phi = (M_\Phi^\diamond)^*, \quad M_\Phi^\diamond = (C_\Phi)^*, \quad \Phi \in \mathcal{W}^*,$$

where $M_\Phi^\diamond \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ is the *Wick multiplication operator* defined by

$$M_\Phi^\diamond \Psi = \Phi \diamond \Psi, \quad \Psi \in \mathcal{W}^*.$$ 

In some literatures, the “convolution product” of $\Phi, \Psi \in \mathcal{W}^*$ is defined by

$$\langle \langle \Phi \star \Psi, \phi \rangle \rangle = \langle \langle \Psi, C_\Phi \phi \rangle \rangle.$$

Using $C_\Phi = (M_\Phi^\diamond)^*$ we see that

$$\langle \langle \Psi, C_\Phi \phi \rangle \rangle = \langle \langle M_\Phi^\diamond \Psi, \phi \rangle \rangle = \langle \langle \Phi \diamond \Psi, \phi \rangle \rangle$$

Therefore, the convolution product $= \text{the Wick product}: \Phi \star \Psi \equiv \Phi \diamond \Psi$.
2. Quantum White Noise Derivatives
2.1. Motivation

Hida’s idea of white noise functional:
\[ F = F(W_t ; t \in T), \text{ where } \{W_t\} \text{ plays a role of orthogonal coordinate system} \]

Our idea

For a white noise operator
\[ \Xi = \Xi(a_s, a^*_t ; s, t \in T) \]
we should like to define the derivatives with respect to \( a_s \) and \( a^*_t \):
\[ \frac{\delta \Xi}{\delta a_s} \text{ and } \frac{\delta \Xi}{\delta a^*_t} \]

Expected properties:
\[ \frac{\delta}{\delta a_s} \int f(t) a_t dt = f(s) I \]
\[ \frac{\delta}{\delta a_s} \int f(s, t) a_s a_t ds dt = \int f(s, t) a_t dt + \int f(t, s) a_t dt \]
\[ \frac{\delta}{\delta a^*_t} \int f(s, t) a_s a^*_t ds dt = \int f(s, t) a_s ds \]
2.2. Definition

**Definition (Ji–O. Sem. et Congres 16 (2008))**

For $\Xi \in \mathcal{L}((E), (E)^*)$ and $\zeta \in E$ we define $D^{\pm}_\zeta \Xi \in \mathcal{L}((E), (E)^*)$ by

$$D^+_\zeta \Xi = [a(\zeta), \Xi], \quad D^-_\zeta \Xi = -[a^*(\zeta), \Xi].$$

These are called the *creation derivative* and *annihilation derivative* of $\Xi$, respectively. Both together are called the *quantum white noise derivatives*.

**Note:** For $\zeta \in E$, both

$$a(\zeta) = \Xi_{0,1}(\zeta) = \int_T \zeta(t) a_t \, dt, \quad a^*(\zeta) = \Xi_{1,0}(\zeta) = \int_T \zeta(t) a^*_t \, dt,$$

belong to $\mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

**Some properties:**

1. $(D^+_\zeta \Xi)^* = D^-_\zeta (\Xi^*)$ and $(D^-_\zeta \Xi)^* = D^+_\zeta (\Xi^*)$.
2. $D^{\pm}_\zeta$ is a continuous linear map from $\mathcal{L}((E), (E)^*)$ into itself.
3. Moreover, $(\zeta, \Xi) \mapsto D^{\pm}_\zeta \Xi$ is a continuous bilinear map from $E \times \mathcal{L}((E), (E)^*)$ into $\mathcal{L}((E), (E)^*)$. 
2.3. Examples

(1) The generalized Gross Laplacian associated with $S$ is defined by

$$\Delta_G(S) = \Xi_{0,2}(\tau_S) = \int_{T \times T} \tau_S(s, t) a_s a_t \, ds \, dt,$$

where $S \in \mathcal{L}(E, E^*)$ and $\tau = \tau_S \in (E \otimes E)^*$ are related as

$$S\xi(s) = \int_T \tau_S(s, t) \xi(t) \, dt.$$

Then, $\Delta_G(S) \in \mathcal{L}((E), (E))$ and

$$D_\zeta^+ \Delta_G(S) = 0, \quad D_\zeta^- \Delta_G(S) = a(S\zeta) + a(S^*\zeta).$$

In fact, since

$$D_t^- \Delta_G(S) = \int_T \tau_S(s, t) a_s \, ds + \int_T \tau_S(t, s) a_s \, ds,$$

we have

$$D_\zeta^- \Delta_G(S) = \int_{T \times T} \tau_S(s, t) a_s \zeta(t) \, ds \, dt + \int_{T \times T} \tau_S(t, s) a_s \zeta(t) \, ds \, dt$$

$$= \int_T S\zeta(s) a_s \, ds + \int_T S^*\zeta(s) a_s \, ds = a(S\zeta) + a(S^*\zeta).$$
### 2.3. Examples (cont)

(2) The adjoint of $\Delta_G(S) \in \mathcal{L}((E)^*, (E)^*)$ is given by

$$\Delta^*_G(S) = \Xi_{2,0}(\tau_S) = \int_{T \times T} \tau_S(s,t)a^*_s a^*_t \, dsdt$$

The quantum white noise derivatives are given by

$$D^-_\zeta \Delta^*_G(S) = 0, \quad D^+_\zeta \Delta^*_G(S) = a^*(S\zeta) + a^*(S^*\zeta)$$

(3) The *conservation operator* associated with $S$ is defined by

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{T \times T} \tau_S(s,t)a^*_s a_t \, dsdt$$

In general, $\Lambda(S) \in \mathcal{L}((E), (E)^*)$.

The quantum white noise derivatives are given by

$$D^-_\zeta \Lambda(S) = a^*(S\zeta), \quad D^+_\zeta \Lambda(S) = a(S^*\zeta).$$
2.4. Wick Derivations

\((\mathcal{L}((E), (E)^*), \diamond)\) is a commutative algebra.

**Definition (Wick derivation)**

A continuous linear map \(D : \mathcal{L}((E), (E)^*) \to \mathcal{L}((E), (E)^*)\) is called a Wick derivation if

\[
D(\Xi_1 \diamond \Xi_2) = (D\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (D\Xi_2), \quad \Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*).
\]

**Theorem (Ji–O. JMP 51 (2010))**

The creation and annihilation derivatives \(D_\zeta^\pm\) are Wick derivations for any \(\zeta \in E\).

▶ Wick derivations for white noise functions [Chung–Chung JKMS 33 (1996)].

**Theorem (Ji–O. JMP 51 (2010))**

A general Wick derivation \(D\) is expressed in the form:

\[
D = \int_T F(t) \diamond D_t^+ \, dt + \int_T G(t) \diamond D_t^- \, dt,
\]

where \(F, G \in E \otimes \mathcal{L}((E), (E)^*)\).
3. Wick Type Differential Equations for White Noise Operators
3.1. A General Result


Let $\mathcal{D} : \mathcal{L}((E), (E)^*) \rightarrow \mathcal{L}((E), (E)^*)$ be a Wick derivation and $G, F \in \mathcal{L}((E), (E)^*)$. Assume that there exist $Y, Z \in \mathcal{L}((E), (E)^*)$ satisfying

(i) $\mathcal{D}Y = G$;

(ii) $\text{wexp } Y \in \mathcal{L}((E), (E)^*)$;

(iii) $\mathcal{D}Z = F \diamond \text{wexp } (-Y)$.

Then a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ is a solution to

$$\mathcal{D}\Xi = G \diamond \Xi + F$$

if and only if $\Xi$ is of the form:

$$\Xi = (Z + C) \diamond \text{wexp } Y$$

with a white noise operator $C \in \mathcal{L}((E), (E)^*)$ satisfying $\mathcal{D}C = 0$.

$$\text{wexp } Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n}, \quad Y \in \mathcal{L}((E), (E)^*),$$
Let us consider the differential equation:

\[ D_\zeta \Xi = 2a(\zeta) \diamond \Xi, \quad \zeta \in E. \]  \hspace{1cm} (3)

Apply our general result (the previous Theorem).

1. We need to find \( Y \in \mathcal{L}( (E), (E)^* ) \) satisfying \( D_\zeta Y = 2a(\zeta) \).
2. In fact, \( Y = \Delta_G \) is a solution.
3. Moreover, it is easily verified that \( \text{wexp} \Delta_G \) is defined in \( \mathcal{L}( (E), (E) ) \).
4. Then, a general solution to (3) is of the form:

\[ \Xi = (\text{wexp} \Delta_G) \diamond F, \]

where \( D_\zeta F = 0 \) for all \( \zeta \in E \).
Now we consider the differential equation:

\[
\begin{cases}
D^-\xi = 2a(\zeta)\diamond \xi, & \zeta \in E, \\
D^+\xi = 0.
\end{cases}
\]  

By the previous Example the solution is of the form:

\[\xi = (\text{wexp } \Delta G) \diamond F, \quad D^-\xi F = 0 \text{ for all } \zeta \in E.\]

We need only to find additional conditions for \( F \) satisfying \( D^+\xi = 0 \).

Noting that \( D^+\Delta G = 0 \), we have

\[D^+\xi = (\text{wexp } \Delta G) \diamond D^+\xi F = 0.\]

Hence \( D^+ F = 0 \) for all \( \zeta \in E \), so \( F \) is a scalar operator (Example (1)).

Consequently, the solution to (4) is of the form:

\[\xi = C \text{ wexp } \Delta G, \quad C \in \mathbb{C}.\]
4. Applications
4.1. Quantum Martingales

Theorem (Ji JFA 201 (2003), also Parthasarathy–Sinha JFA 67 (1986))

A regular quantum martingale \( \{M_t\}_{t \in \mathbb{R}^+} \subset \mathcal{L}(\mathcal{G}_p(\mathbb{R}^+), \mathcal{G}_q(\mathbb{R}^+)) \) admits an integral representation:

\[
M_t = \lambda I + \int_0^t (E_dA + F_dA^* + Gd\Lambda),
\]

where \( \{E_t\}, \{F_t\}, \{G_t\} \) in \( \mathcal{L}(\mathcal{G}_p(\mathbb{R}^+), \mathcal{G}_q(\mathbb{R}^+)) \) are adapted processes and \( \lambda \in \mathbb{C} \).

Theorem (Ji–O. CMP 286 (2009))

The integrands of \( M_t \) is obtained by

\[
E_s = D_s^- \left[ M_s - \int_0^s a_u^* (D_u^+ M_u) du \right],
\]

\[
F_s = D_s^+ \left[ M_s - \int_0^s (D_u^- M_u) a_u du \right],
\]

\[
G_s = D_s^+ \left[ \int_0^s \left\{ D_u^- \left( M_u - \int_0^u E_v a_v dv - \int_0^u a_v^* F_v dv \right) \right\} du \right].
\]
4.1. Quantum Martingales (cont)

Quantum stochastic integrals of Itô type \[ \int_0^t E dA, \int_0^t E dA^*, \int_0^t E d\Lambda \]

\[ \implies \text{Quantum Hitsuda Skorohod integrals } \delta^- (\Xi), \delta^+ (\Xi), \delta^\circ (\Xi) \]

e.g., the creation gradient is defined:

\[ \nabla^+ : \mathcal{L}((E), D) \xrightarrow{\cong} D \otimes (E)^* \xrightarrow{\nabla \otimes I} L^2(\mathbb{R}, \Gamma(H)) \otimes (E)^* \]

\[ \cong L^2(\mathbb{R}, \Gamma(H) \otimes (E)^*) \cong L^2(\mathbb{R}, \mathcal{L}((E), \Gamma(H))) \]

Then, the creation integral is defined by

\[ \delta^+ = (\nabla^+)^*: L^2(\mathbb{R}, \mathcal{L}((E)^*, \Gamma(H)))) \rightarrow \mathcal{L}((E)^*, D^*) \]

\[ \delta^+ (\Xi) = (\text{non-adapted extension of}) \int \Xi(s) dA_s^* ds \]

Quantum white noise derivatives of quantum stochastic integrals, e.g.,

\[ D^+_\zeta (\delta^+ (\Xi)) = \delta^+ (D^+_\zeta \Xi) + \int_{\mathbb{R}} \zeta(t) \Xi(t) dt \]
Let $S, T \in \mathcal{L}(E, E)$ and consider transformed annihilation and creation operators:

$$b(\zeta) = a(S\zeta) + a^*(T\zeta), \quad b^*(\zeta) = a^*(S\zeta) + a(T\zeta),$$

where $\zeta \in E$. We know that $b(\zeta), b^*(\zeta) \in \mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

**The implementation problem**

is to find a white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfying

$$
\begin{align*}
(U) : & (E) \xrightarrow{U} (E)^* \\
& a(\zeta) \downarrow \quad b(\zeta) \\
(U) : & (E) \xrightarrow{U} (E)^* \\
& \downarrow \quad \downarrow
\end{align*}
$$

$$
\begin{align*}
(U) : & (E) \xrightarrow{U} (E)^* \\
& a^*(\zeta) \downarrow \quad b^*(\zeta) \\
(U) : & (E) \xrightarrow{U} (E)^* \\
& \downarrow \quad \downarrow
\end{align*}
$$

**Key observation**

$$
\begin{align*}
Ua(\zeta) = b(\zeta)U & \iff \quad D_{S\zeta}^+ U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U, \\
Ua^*(\zeta) = b^*(\zeta)U & \iff \quad (D_{\zeta}^- - D_{T\zeta}^+)U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U.
\end{align*}
$$
Proof of key observation

\[ Ua(\zeta) = b(\zeta)U \]
\[ = (a(S\zeta) + a^*(T\zeta))U \]
\[ = [a(S\zeta), U] + Ua(S\zeta) + a^*(T\zeta)U \]
\[ = D_{S\zeta}^+ U + Ua(S\zeta) + a^*(T\zeta)U, \]

Hence

\[ D_{S\zeta}^+ U = Ua(\zeta) - Ua(S\zeta) - a^*(T\zeta)U \]
\[ = Ua(\zeta - S\zeta) - a^*(T\zeta)U \]
\[ = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \]

**Remark** If \( b(\zeta) \) is replaced with \( b(\zeta) = \int_T S\zeta(t)a_t^m \, dt + \int_T T\zeta(t)(a_t^*)^n \, dt \), we need to define quantum white noise derivatives with respect to the higher powers of quantum white noise:

\[ D_{S\zeta}^+ U = \left[ \int_T S\zeta(t)a_t^m \, dt, U \right]. \]

But this is not a Wick derivation.
4.2. The Implementation Problem for CCR (cont)

**Theorem (Ji–O. JMP 51 (2010))**

Assume the following conditions:

(i) $S$ is invertible;

(ii) $T^* S = S^* T \iff [b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$;

(iii) $S^* S - T^* T = I \iff [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle$;

(iv) $ST^* = TS^*$.

A white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfies the following intertwining properties:

$$Ua(\zeta) = b(\zeta)U, \quad Ua^*(\zeta) = b^*(\zeta)U, \quad \zeta \in E,$$

if and only if $U$ is of the form:

$$U = C \; \text{wexp} \left\{-\frac{1}{2} \Delta_G^* (TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2} \Delta_G (S^{-1}T)\right\}$$

$$= C e^{-\frac{1}{2} \Delta_G^* (TS^{-1})} \Gamma((S^{-1})^*) e^{\frac{1}{2} \Delta_G (S^{-1}T)},$$

where $C \in \mathbb{C}$. This is a (generalization of) Bogolubov transformation.
4.3. The Implementation Problem for CCR — Slightly Generalized

\[ b(\zeta) = a(S\zeta) + a^*(T\zeta) + \langle k, \zeta \rangle, \quad b^*(\zeta) = a^*(S\zeta) + a(T\zeta) + \langle k, \zeta \rangle \]

\[ Ua(\zeta) = b(\zeta)U \iff D_{S\zeta}^+ U = [a(\zeta - S\zeta) - a^*(T\zeta) - \langle k, \zeta \rangle] \diamond U \tag{1} \]

\[ Ua^*(\zeta) = b^*(\zeta)U \]

\[ \iff (D_\zeta^- - D_{T\zeta}^+) U = [a^*(S\zeta - \zeta) + a(T\zeta) + \langle k, \zeta \rangle] \diamond U \tag{2} \]

\[ U = \text{wexp} \left\{-a^*((S^{-1})^*k) - \frac{1}{2}\Delta^*_G(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F, \]

where \( F \in \mathcal{L}((E), (E)^*) \) fulfills \( D_{\zeta}^+ F = 0 \) for all \( \zeta \in E \).

\[ U = \text{wexp} \left\{-\frac{1}{2}\Delta^*_G(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_G(S^{-1}T) + a(k)\right\} \diamond G, \]

where \( G \in \mathcal{L}((E), (E)^*) \) is an arbitrary white noise operator satisfying

\[ (D_{\zeta}^- - D_{T\zeta}^+) G = 0 \quad \text{for all} \quad \zeta \in E. \]
4.4. Finding a Normal-Ordered Form of White Noise Operators

Normal-ordered form

(an operator on Fock space) = \( \sum \) (creation operators)(annihilation operators)

= : (an operator on Fock space) :

Example 1 (CCR)

\[ A(f)A(g)^* = A(g)^*A(f) + \langle f, g \rangle \]

Example 2 (repeated application of CCR)

\[ A(f)(A^*(g))^n = (A^*(g))^nA(f) + n\langle f, g \rangle (A^*(g))^{n-1} \]

\[ e^{A(f)}e^{A^*(g)} = e^{\langle f, g \rangle}e^{A^*(g)}e^{A(f)} \]

Question: What about

\[ e^{\Delta_G(S)}e^{\Delta^*_G(T)} \]
The normal-ordered form of $e^\Delta_G(S) e^{\Delta^*_G(T)}$ ($S = S^*, T = T^*$) is given by

$$e^\Delta_G(S) e^{\Delta^*_G(T)} = C e^{\Delta^*_G(T(I-4ST)^{-1})} \Gamma((I-4ST)^{-1}) e^\Delta_G(S(I-4ST)^{-1}).$$

Set $\Xi = e^\Delta_G(S) e^{\Delta^*_G(T)}$ and derive a Wick type differential equation.

$$D^+_\zeta \Xi = e^\Delta_G(S) \cdot 2a^*(T\zeta) \cdot e^{\Delta^*_G(T)}$$

$$= \left\{ -2[a^*(T\zeta), e^\Delta_G(S)] + 2a^*(T\zeta) e^\Delta_G(S) \right\} e^{\Delta^*_G(T)}$$

$$= \left\{ 2D^-_{T\zeta} e^\Delta_G(S) + 2a^*(T\zeta) e^\Delta_G(S) \right\} e^{\Delta^*_G(T)}$$

$$= 2 \cdot 2a(ST\zeta) e^\Delta_G(S) \cdot e^{\Delta^*_G(T)} + 2a^*(T\zeta) \Xi$$

$$= 4a(ST\zeta) \Xi + 2a^*(T\zeta) \Xi$$

$$= 4D^+_{ST\zeta} \Xi + 4\Xi a(ST\zeta) + 2a^*(T\zeta) \Xi$$

Hence

$$D^+_{(I-4ST)\zeta} \Xi = (4a(ST\zeta) + 2a^*(T\zeta)) \diamond \Xi$$
Assume that $I - 4ST$ is invertible. Then we obtain

$$D_\zeta^+ \Xi = \left\{ a(((I - 4ST)^{-1} - I)\zeta) + 2a^*(T(I - 4ST)^{-1}\zeta) \right\} \diamond \Xi \quad (1)$$

Similarly,

$$D_\zeta^- \Xi = \left\{ a^*((((I - 4TS)^{-1} - I)\zeta) + 2a(S(I - 4TS)^{-1}\zeta) \right\} \diamond \Xi \quad (2)$$

General solutions to (1) and (2) are obtained by our method mentioned before:

$$\Xi = \exp \Delta^*_G(T(I - 4ST)^{-1}) \diamond \exp \Lambda((I - 4ST)^{-1} - I) \diamond \text{(annihilations)}$$

$$= \text{(creations)} \diamond \exp \Lambda((I - 4TS)^{-1} - I) \diamond \exp \Delta_G(S(I - 4TS)^{-1})$$

Assuming that $ST = TS$, we obtain

$$\Xi = C \exp \Delta^*_G(T(I - 4ST)^{-1})$$

$$\diamond \exp \Lambda((I - 4ST)^{-1} - I) \diamond \exp \Delta_G(S(I - 4ST)^{-1}).$$

Consequently,

$$e^{\Delta_G(S)} e^{\Delta^*_G(T)} = Ce^{\Delta^*_G(T(I - 4ST)^{-1})} \Gamma((I - 4ST)^{-1}) e^{\Delta_G(S(I - 4ST)^{-1})},$$

where the constant $C$ is obtained from vacuum expectation ($C = \det(1 - 4ST)^{-1/2}$).
4.5. Some Questions

1. So far a linear equation: \( D \Xi = G \diamond \Xi + F \) for characterization of white noise operators.

2. Extension to higher order linear equations?

3. Extension to non-linear equations?

4. Structure of the \( * \)-algebra generated by \( \{D_\zeta^+, D_\zeta^-\} \) and their adjoints?

5. (some challenge) Using higher powers of quantum white noise?

\[
c(\zeta) = \int_T \zeta(t) a_t^m dt, \quad c^*(\zeta) = \int_T \zeta(t) a_t^{*m} dt
\]

Can define

\[
D_\zeta^+(\Xi) = [c(\zeta), \Xi], \quad D_\zeta^-(\Xi) = -[c^*(\zeta), \Xi]
\]

Note: whenever well-defined, these are derivations with respect to the usual product (composition) of white noise operators, but are not Wick derivations.

6. (some physical interpretation) generalizing the Bogolubov transformation between \( (E)^* \) (beyond \( \Gamma(H) \))? then unitarity?