Generalized fraction evolution equations with Fractional Gross Laplacian

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Preliminaries:
- Space of entire functions with growth condition
- Characterisation theorem

Gross Laplacian
- Gross Laplacian
- $K$-Gross Laplacian

Power of the Gross Laplacian
- Definition
- Heat-type Cauchy problem associated to the powers of the Gross Laplacian

Fractional Gross Laplacian
- Mittag-Leffler function
- Fractional Gross Laplacian

Fractional diffusion Equations
- Solution of a linear stochastic differential equations
- Riemann-Liouville time fractional diffusion Equations
Let $N$ the complex nuclear Fréchet space with topology given by an increasing family $\{\| \cdot \|_p, p \in \mathbb{N}\}$:

\[
N = \bigcap_{p \in \mathbb{N}} N_p = \bigcap_{p \in \mathbb{N}} \overline{N}_{\| \cdot \|_p}, \quad N' = \bigcup_{p \in \mathbb{N}} N_{-p} = \bigcup_{p \in \mathbb{N}} N'_{-p}.
\]
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Let $\theta : [0, \infty] \to [0, \infty]$ be a Young function, i.e., continuous, convex, increasing, $\theta(0) = 0$ and

$$\lim_{x \to +\infty} \frac{\theta(x)}{x} = +\infty.$$
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Space of entires functions with growth condition

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\[
\lim_{x \to +\infty} \frac{\theta(x)}{x} = +\infty.
\]

- \( \theta^* \) the conjugate function defined by:

\[
\theta^*(x) := \sup_{t \geq 0} (tx - \theta(t)), \quad \forall x > 0.
\]
**Definition**

1. \( H(N) \) the set of holomorphic functions on \( N \).
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2. $\|f\|_{(\theta,p,m)} = \sup_{z \in \mathbb{N}_p} \{|f(z)| e^{-\theta(m|z|_p)}\}$, for $m \in \mathbb{N}$ and $n > 0$. 
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\[ \sim \Rightarrow \mathcal{F}_\theta(N') = \lim_{\substack{p \to \infty \\quad m \downarrow 0}} \text{Exp}(N_{-p}, \theta, m), \quad (1) \]
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$$\sim \mathcal{F}_\theta(N') = \operatorname{proj \lim}_{p \to \infty} \text{Exp}(N_{-p}, \theta, m),$$  \hspace{1cm} (1)

$$\sim \mathcal{G}_\theta(N) = \operatorname{ind \lim}_{p \to \infty} \operatorname{Exp}(N_p \times N_q, \theta, m).$$  \hspace{1cm} (2)
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4. $\mathcal{F}_\theta^*(N')$ the strong dual topology of $\mathcal{F}_\theta(N')$. 

S. Horrigue
Fractional differential equations
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Space of entires functions with growth condition
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\[ (\theta)_n := \inf_{r > 0} \frac{e^{\theta(r)}}{r^n}, \quad n \in \mathbb{N}. \]
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$$\|\|\| \vec{\varphi} \|\|_{(\theta,p,m)}^2 = \sum_{n \in \mathbb{N}} (\theta)^{-2} m^{-n} |\varphi_n|_p^2$$  \hspace{1cm} (3)
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For all \( \varphi = (\varphi_n)_{n \in \mathbb{N}} \) the formal series such that \( \varphi_n \in (N)_p \), we put

\[ \rightarrow \ |\| \varphi \rangle |^2_{(\theta,p,m)} = \sum_{n \in \mathbb{N}} (\theta)_n^{-2} m^{-n} |\varphi_n|_p^2 \]  \( \rightarrow \) (3)

\[ \Rightarrow F_{(\theta,m)}((N)_p) = \{ \varphi ; |\| \varphi \rangle |^2_{(\theta,p,m)} < \infty \} \]
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For all $\vec{\varphi} = (\varphi_n)_{n \in \mathbb{N}}$ the formal series such that $\varphi_n \in (N)_{\mathbb{P}}{^n}$, we put

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$$\Rightarrow F_{(\theta, m)}((N)_{\mathbb{P}}) = \{ \vec{\varphi}; ||\vec{\varphi}||_{(\theta, p, m)} < \infty \}$$

$$\Rightarrow F_{\theta}(N) = \lim_{{p \to \infty \atop m \downarrow 0}} F_{(\theta, m)}((N)_{\mathbb{P}}).$$
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$$\rightarrow G_{\theta}(\mathbb{N}') = \text{ind lim} \ G_{(\theta,m)}((\mathbb{N})_{p}),$$

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For all $\vec{\Phi} = (\Phi_n)_{n \in \mathbb{N}}$ the formal series such that $\Phi_n \in (N)^\otimes n$, we put
\[ ||| \vec{\Phi} |||^2_{(\theta, -p, m)} = \sum_{n \in \mathbb{N}} [n! (\theta)_n]^2 m^n |\Phi_n|_{-p}^2, \quad (5) \]

\[ \Phi \mapsto G(\theta, m)((N)_p) = \{ \vec{\Phi} = (\Phi_n)_{n \in \mathbb{N}}; ||| \vec{\Phi} |||_{(\theta, -p, m)} < \infty \} \]

\[ G_\theta(N') = \text{ind lim}_{p \to \infty} \text{ind lim}_{m \to 0} G(\theta, m)((N)_p), \quad (6) \]

which is the topological dual of $F_\theta(N)$ with the dual pairing:
\[ \langle \vec{\Phi}, \vec{\varphi} \rangle = \sum_{n \in \mathbb{N}} n! \langle \Phi_n, \varphi_n \rangle. \]
Taylor transformation

Taylor transformation $\mathcal{T}$ is defined by

$$\mathcal{T} : f \in \mathcal{H}(\mathbb{N}) \mapsto \vec{f} = \left( \frac{1}{n!} f^n(0) \right).$$  \hspace{1cm} (7)
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**Theorem**

We have the following isomorphisms topology

$$\mathcal{T} : F_\theta(N') \rightarrow F_\theta(N)$$

(8)

$$\mathcal{G}_\theta(N) \rightarrow \mathcal{G}_\theta(N').$$

(9)
For a generic function $f(t)$ the usual Laplace transform is given by

$$Lf(s) := \int_0^\infty e^{-st} f(t) dt, \quad s \in \mathbb{C}$$
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★ Let $\xi \in \mathbb{N}$. We define the exponential function by

$$e^{\xi}(x) = \sum_{n \in \mathbb{N}} \langle x^\otimes n, \frac{\xi^\otimes n}{n!} \rangle, \quad x \in \mathbb{N}'.$$  \hspace{1cm} (10)
Laplace transformation

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\]  

\[
\rightsquigarrow \text{The Laplace transformation}
\]

\[
\mathcal{L} : \Phi \in \mathcal{F}_\theta^*(\mathbb{N}') \mapsto \mathcal{L}(\Phi)(\xi) = \langle \Phi, e^\xi \rangle.
\]
Theorem

We have the following isomorphisms topology

\[ \mathcal{L} : \mathcal{F}_\theta^*(N') \rightarrow \mathcal{G}_\theta^*(N). \]  

(12)
Symbol transformation

\[
\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta^*(N')) : \text{space of linear continuous operators on } \mathcal{F}_\theta(N') \text{ into } \mathcal{F}_\theta^*(N').
\]
Symbol transformation

★ \( L(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N')) \) : space of linear continuous operators on \( \mathcal{F}_\theta(N') \) into \( \mathcal{F}_\theta^*(N') \).

\( \sim \) Symbol map

\[ \sigma : \Xi \in L(\mathcal{F}_\theta(N'),\mathcal{F}_\theta^*(N')) \mapsto \sigma(\Xi)(\xi,\eta) = \langle \Xi e_\xi, e_\eta \rangle = \hat{\Xi}^K(\xi,\eta) \]
Convolution product

Convolution product:
♦ of distribution $\Phi \in \mathcal{F}_\theta^*(N')$ and test function $\varphi \in \mathcal{F}_\theta(N')$ is
defined by

$$
\langle \langle \Phi \ast \varphi, \tau \rangle \rangle = \langle \langle \Phi, \varphi(z+.) \rangle \rangle,
$$
(13)

♦ of two distributions $\Phi, \Psi \in \mathcal{F}_\theta^*(N')$ is defined by

$$
\langle \langle \Phi \ast \Psi, \varphi \rangle \rangle = \langle \langle \Phi, \Psi \ast \varphi \rangle \rangle.
$$
(14)

♦ of two operators $\Xi_1, \Xi_2 \in L(\mathcal{F}_\theta^*(N'), \mathcal{F}_\theta(N'))$ is defined by

$$
\sigma(\Xi_1 \ast \Xi_2) = \sigma(\Xi_1) \sigma(\Xi_2).
$$
(15)
Convolution product

\[ \Phi \ast \varphi(z, t) = \langle\langle \Phi, \tau_z \varphi \rangle\rangle = \langle\langle \Phi, \varphi(z + .) \rangle\rangle, \quad (13) \]
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\sigma(\Xi_1 \ast \Xi_2) = \sigma(\Xi_1)\sigma(\Xi_2).
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(15)
Gross Laplacian acting on the space of entire functions in one infinite dimensional variable

- L. Gross defined the Laplacian in the Wiener space \((H, B)\) by
\[
\Delta_G F(x) = \text{trace} F''(x), \quad \forall F \in C^2(H).
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- Let \(\tau\) the trace operator defined on \(\mathcal{N} \otimes^2\) as follows
  \[ \langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \quad \xi, \eta \in \mathcal{N}. \]
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_M. Ben Chrouda, M. El Oued, and H. Ouerdiane,_

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**M. Ben Chrouda, M. El Oued, and H. Ouerdiane,**


\[ \forall \varphi \in \mathcal{F}_0(N') \]

S. Horrigue  Fractional differential equations
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\[ \Rightarrow \Delta_G \varphi = \mathcal{T} \ast \varphi, \quad \forall \varphi \in \mathcal{F}_\theta(N') \]

where the distribution \(\mathcal{T} = (\mathcal{T}_n)_{n \in \mathbb{N}} = (0, 0, \tau, 0, \cdots).\)
Gross Laplacian acting on the space of entire functions in one infinite dimensional variable

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M. Ben Chrouda, M. El Oued, and H. Ouerdiane,


\[\Rightarrow \Delta_G \varphi = \mathcal{T} \ast \varphi, \quad \forall \varphi \in \mathcal{F}_\theta(N')\]
where the distribution \(\mathcal{T} = (\mathcal{T}_n)_{n \in \mathbb{N}} = (0, 0, \tau, 0, \cdots)\).

\[\Rightarrow \Delta_G \Phi = \mathcal{T} \ast \Phi, \quad \Phi \in \mathcal{F}_{\theta}^*(N').\]
Let $K \in L(N, N')$. We denote by $\tau_K$ the kernel associated to $K$ in $(N \otimes^2)'$ which is defined by

$$\langle \tau_K, \xi \otimes \eta \rangle = \langle K\xi, \eta \rangle, \quad \forall (\xi, \eta) \in N \otimes^2.$$
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$$\langle \tau_K, \xi \otimes \eta \rangle = \langle K \xi, \eta \rangle, \quad \forall (\xi, \eta) \in N \otimes^2.$$  

$T_K = ((T_K)_n)_{n \in \mathbb{N}} \in \mathcal{F}_\theta^*(N')$ is defined by

$$(T_K)_n = (T_K)_n = \begin{cases} \tau_K, & n = 2, \\ 0, & \text{otherwise.} \end{cases}$$
The $K$-Gross Laplacian is defined on
The $K$-Gross Laplacian is defined on $\mathcal{F}_\theta(N')$ into itself by

$$\Delta_G(K)\varphi(x) = T_K \ast \varphi(x)$$

$$= \sum_{n \in \mathbb{N}} \langle x^\otimes, (n + 2)(n + 1) \langle \tau_K, \varphi_{n+2} \rangle, \rangle,$$

for all $\varphi = (\varphi_n)_{n \in \mathbb{N}} \in \mathcal{F}_\theta(N')$ and $x \in N'$.

We assume that the Young function $\theta$ satisfies the following condition

$$\limsup_{x \to \infty} \frac{\theta(x)}{x^2} < +\infty.$$
The $K$-Gross Laplacian is defined on $\mathcal{F}_\theta(N')$ into itself by

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for all $\varphi = (\varphi_n)_{n \in \mathbb{N}} \in \mathcal{F}_\theta(N')$ and $x \in N'$. We assume that the Young function $\theta$ satisfies the following condition

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on generalized functions space $\mathcal{F}_\theta^*(N')$

$$\Delta_G(K)\psi = T_K \ast \psi, \psi \in \mathcal{F}_\theta^*(N').$$
Proposition

The symbol of the $K$-Gross Laplacian is given by:

$$
\sigma(\Delta_G^K(\xi_1,\xi_2)) = \langle K\xi_1,\xi_1 \rangle \exp(\langle \xi_1,\xi_2 \rangle),
$$
for all $(\xi_1,\xi_2) \in N' \otimes 2$.
Proposition

The symbol of the $K$-Gross Laplacian is given by:

$$\sigma(\Delta_G(K))(\xi_1, \xi_2) = (\langle K \xi_1, \xi_1 \rangle) \exp(\langle \xi_1, \xi_2 \rangle),$$

for all $(\xi_1, \xi_2) \in N^\otimes 2$. 

(16)
Power of the Gross Laplacian


**Proposition**

For every positive integer $p$ we have

\[ \Delta^p_G \Psi = (T^*p) \ast \Psi, \quad \Psi \in \mathcal{F}_\theta^*(N'). \]  

(17)

Moreover, the associated generalized function $T^{*p}$ with $\Delta^p_G$ is given by

\[ T^{*p} \rightarrow (0, \ldots, 0, \tau \otimes^{2p-position}, 0\ldots). \]  

(18)
Proposition

The symbol of the $p$-power Gross Laplacian is given by:

$$\sigma(\Delta^p_G)(\xi, \eta) = \langle \xi, \xi \rangle^p \exp(\langle \xi, \eta \rangle), \ \forall (\xi, \eta) \in \mathbb{N}^\otimes 2.$$
Proposition

For every distribution $\Phi \in \mathcal{F}_\theta^*(N')$, the functional $e^*\Phi$ is defined by

$$\hat{e}^*\Phi = e^\Phi$$  \hspace{1cm} (19)

Moreover, the functional $e^*\Phi$ belongs to $\mathcal{F}_{(e^\theta^*)^*}(N')$.

Let $I \subset \mathbb{R}$ be an interval containing the origin. Consider a family $\{\Phi_t; t \in I\}$ of distributions in $\mathcal{F}_\theta^*(N')$. Consider the following initial value problem :

$$\left\{\begin{array}{l}
\frac{d\Phi(t)}{dt} = \frac{1}{2}(-1)^{p+1} \Delta^p_G \Phi(t) + \Psi(t) \\
\Phi(0) = \Phi_0 \in \mathcal{F}_\theta^*(N').
\end{array}\right.$$ \hspace{1cm} (20)

S. Horrigue Fractional differential equations
Theorem

Let \( \theta \) be a Young function such that \( \lim_{r \to \infty} \frac{\theta(r)}{r^2} < \infty \). Let \( \Phi_0 \in \mathcal{F}_\theta^* (N') \). Then the power Gross heat equation (25), perturbed by the potential \( \Psi_t \), has a unique solution in \( \mathcal{F}(e^{\theta^*})(N') \) given by

\[
\Phi(t) = \Phi_0 * \left( e^{\frac{t}{2}} \mathcal{T}^* \right)^p + \int_0^t e^{\frac{t-s}{2}} \mathcal{T}^* \Psi(s) \, ds. \tag{21}
\]
We can further rewrite the solution in equation (25) in another form. For \( t > 0 \), define the distribution \( \mu_{t,p} \) by its Laplace transform

\[
\widehat{\mu_{t,p}}(\xi) = \exp \left[ \frac{(-1)^{p+1} t}{2} \langle \xi, \xi \rangle^p \right], \quad \xi \in \mathbb{N}.
\] (22)

Recall that the duality theorem states that the Laplace transform is a topological isomorphism from \( \mathcal{F}_{\theta}^*(\mathbb{N}') \) onto \( \mathcal{G}_{\theta^*}(\mathbb{N}) \). Hence Equation (22) implies that \( \mu_{t,p}, \ t > 0 \), are generalized functions in the space \( \mathcal{F}_{\theta}(\mathbb{N}') \) with the Young function given by

\[
\theta(x) = x^{\frac{2p}{2p-1}}, \quad x \geq 0.
\]

**Corollary**

Let \( \Phi_0 \in \mathcal{F}_{\theta}(\mathbb{N}') \). Then, the solution \( \Phi_t \) in equation (25) can be rewritten as

\[
\Phi(t) = \Xi_0 * \mu_{t,p} + \int_0^t \mu_{t-s,p} * \Psi(s) ds.
\]
Definition

The Mittag-Leffler function is an entire function defined by the series

\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},
\]

for \( \alpha > 0 \). The more general Mittag-Leffler function

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},
\]

can also be defined for \( \alpha, \beta > 0 \), so that

\[
E_\alpha(z) = E_{\alpha,1}(z).
\]
**Proposition**

Let

\[ N \ni \xi \mapsto E_{\alpha,\beta}(\langle \xi, \xi \rangle) \in \mathbb{C}, \]

is an element of \( G_{\gamma^*}(N) \) where \( \gamma \) is the Young function defined by

\[ \gamma(x) = x^{2-\alpha}, \text{ where } 0 < \alpha < 1. \]

Moreover the corresponding Taylor series is given by

\[ E_{\alpha,\beta}(\langle \xi, \xi \rangle) := \sum_{n=0}^{\infty} \langle E_{\alpha,\beta}^n, \xi \otimes n \rangle, \]

where the kernels \( E_{\alpha,\beta}^n \) are given

\[ \begin{cases} E_{\alpha,\beta}^{2n} = \frac{\tau \otimes n}{\Gamma(\alpha n + \beta)}, \\ E_{\alpha,\beta}^{2n+1} = 0. \end{cases} \]
Let $\alpha$ be a positive fractional.

**Definition**

An operator is called a fractional Gross Laplacian denoted by $\Delta^2_G$ if it satisfies the following property for all $\varphi \in \mathcal{F}_\theta(N')$:

$$\mathcal{L}(\Delta^2_G \varphi)(\xi) = \langle \xi, \xi \rangle^\alpha \hat{\varphi}(\xi) \quad \xi \in N.$$
**Theorem**

The fractional Gross Laplacian $\Delta_{G}^{2\alpha}$ is well defined on $\mathcal{F}_{\theta}(N')$ into itself.

**Proof.**

Using the isomorphism (12), to prove that the fractional Gross Laplacian $\Delta_{G}^{2\alpha}$ is well defined on $\mathcal{F}_{\theta}(N')$ into itself is sufficient to prove that $\mathcal{L}(\Delta_{G}^{2\alpha}) \in \mathcal{G}_{\theta^*}(N')$.
Consider now the following initial value problem:

\[
\begin{cases}
\frac{d}{dt} \Phi(t) = Z(t) \ast \Phi(t) + \Psi(t) \\
\Phi(0) = \Phi_0,
\end{cases}
\]  

(25)

where \(\{Z(t), t \in I\}\) and \(\{\Psi(t), t \in I\}\) are stochastic processes defined on an interval \(I\) into \(\mathcal{F}_\theta(N')\) and \(\Phi_0 \in \mathcal{F}_\theta(N')\).

**Theorem**

The stochastic differential equation (25) has a unique solution in \(\mathcal{F}(e^{\theta \ast})(N')\) given by

\[
\Phi(t) = \Phi_0 \ast e^{\ast} \int_0^t Z(s)ds + \int_0^t e^{\ast} (\int_s^t Z(u)du) \ast \Psi(s)ds.
\]  

(26)
Proof.
Applying the symbol map to the differential equation (25), we obtain an ordinary differential equation given by

\[
\begin{aligned}
\frac{d}{dt} \hat{\Phi}(t) &= \hat{Z}(t)\hat{\Phi}(t) + \hat{\Psi}(t) \\
\hat{\Phi}(0) &= \hat{\Phi}_0 \in \mathcal{F}_\theta^*(N'), \\
\end{aligned}
\]

whose solution is given by

\[
\hat{\Phi}(t) = \hat{\Phi}_0 e^{\int_0^t \hat{Z}(s)ds} + \int_0^t e^{\int_s^t \hat{Z}(u)du} \hat{\Psi}(s)ds.
\]

Then, by Proposition 2.1, \(\hat{\Phi}(t) \in \mathcal{F}^*(e^{\theta^*})^*(N'),\) for all \(t \in I.\) The solution of 27 is given by

\[
\Phi(t) = \Phi_0^*e^{\int_0^t Z(s)ds} + \int_0^t e^*\left(\int_s^t Z(u)du\right)^*\Psi_s ds \in \mathcal{F}(e^{\theta^*})^*(N'), \quad \forall t \in I.
\]
In this part, we consider to the following Riemann-Liouville time fractional diffusion Equations

\[
\begin{cases}
RLD_t^\beta U(t), & = \Delta_G^\alpha U_t + V(t), \quad t > 0; \\
D_t^{\beta-1} U(t)|_{t=0}, & = U_0.
\end{cases}
\]

where \( \Phi \in \mathcal{F}_{\theta}^\prime (N^\prime) \) and \( RL D_t^\beta \) is the Riemann-Liouville operator defined by

\[
RL D_t^\beta U(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{U(\tau)}{(t-\tau)^\beta} d\tau, \quad 0 < \beta < 1
\]

and \( D_t^{\alpha-1} U(t)|_{t=0} \) is given by

\[
D_t^{\beta-1} U(t)|_{t=0} := \lim_{t \downarrow 0} \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{U(\tau)}{(t-\tau)^\beta} d\tau.
\]
Lemma

For any generalized function \( \Psi \in \mathcal{F}'_\theta(N') \) we have

\[
L\left( RLD_\beta \Psi(.) \right)(s) = s^\beta L(\Psi(.))(s) - D_t^{\beta-1} \Psi(t) \big|_{t=0}
\]

Lemma

For any fractional \( \alpha, \beta, s \in \mathbb{C} \) and \( \xi \in N \), such that \( \mathcal{R}(s) > |\langle \xi, \xi \rangle|^{\frac{1}{\alpha}} \), we have

\[
\frac{1}{s^\beta - \langle \xi, \xi \rangle^\alpha} = \langle \xi, \xi \rangle \frac{(\alpha-\beta)(\alpha-1)}{\beta} L(t^{\alpha-1} \Xi_{\alpha,\alpha}(\langle \xi, \xi \rangle^\beta t^\alpha))(s).
\]
Notation

Let $\theta$ and $\gamma$ be two Young functions. Denoted by

$$\liminf_{x \to \infty} \frac{\theta(x)}{\gamma(x)} = c_1,$$

and

$$\limsup_{x \to \infty} \frac{\theta(x)}{\gamma(x)} = c_2.$$
The following theorem gives the existence for the solution of the Riemann-Liouville time fractional diffusion equation.

**Theorem**

Let $U_0 \in \mathcal{F}_{\theta}(N')$, $\gamma_p(x) = x^{\frac{2p}{2p-\alpha}}$ where $p$ is the integral part of $\frac{\alpha}{\beta}$ and $c_1, c_2$ be as in (29),(30). The solution of the fractional diffusion equation in $S_1$ is given by

$$U(t) = t^{\alpha-1} \Psi_{\alpha,\alpha, t} * U_0$$

where $c_1$ and $c_2$ are like introduced above and $\Psi_{\alpha, t}$ satisfies

$$\mathcal{L} \Psi_{\alpha, \beta, t}(\xi) = \Xi_{\alpha, \beta}(\langle \xi, \xi \rangle t^\alpha)$$

Moreover, the solution $U(t)$ satisfies

$$U(t) \in \begin{cases} \mathcal{F}'_{\gamma}(N') & \text{if } c_1 \leq c_2 < \infty \text{ or } c_1 = \infty \\ \mathcal{F}'_{\theta}(N') & \text{if } c_2 = 0. \end{cases}$$
Proof.

\[ RL D^\beta_t U(t) = \Delta^\alpha_G U(t) + V(t) \]

Using the representation of the Gross Laplacian and applying the Laplace transform in the equation of Riemann-liouville we obtain

\[ L(RL D^\beta_t U(.))(s) = s^\beta L(U(.))(s) - U(0) \]
\[ = \tau^\alpha \ast L(U(.))(s) + L(V(.))(s) \]

which implies that

\[ s^\beta L[LU(.)(s)](\xi) - L(U_0)(\xi) = \langle \xi, \xi \rangle^\alpha L(LU(.)(s))(\xi) + L[LV(.)](s)(\xi) \]

We assume that \( f = L[LU(.)(s)](\xi) \). Then, we have

\[ (s^\beta - \langle \xi, \xi \rangle^\alpha) f = L(U_0)(\xi) + L(LV(.))(s)(\xi) \]
Therefore,

\[
f = \frac{\mathcal{L}(U_0)(\xi) + \mathcal{L}(L V(\cdot))(s)(\xi)}{s^\beta - \langle \xi, \xi \rangle^\alpha} \]

\[
= \langle \xi, \xi \rangle^{(\alpha-\beta)(\alpha-1)\beta^{-1}} L \left( t^{\alpha-1} \Xi_{\alpha, \alpha} \left( \langle \xi, \xi \rangle^{\alpha \beta} t^\alpha \right) \right)(s) \left[ \mathcal{L}(U_0)(\xi) + \mathcal{L}(L V(\cdot))(s) \right]
\]

\[
L^{-1}(f) = \mathcal{L}(U(t))(\xi) - \langle \xi, \xi \rangle^{(\alpha-\beta)(\alpha-1)\beta^{-1}} t^{\alpha-1} \Xi_{\alpha, \beta} \left( \langle \xi, \xi \rangle^{\alpha \beta} t^\alpha \right) \mathcal{L}(U_0)(\xi) + \mathcal{L}(V)(\xi)(t) * \langle \xi, \xi \rangle^{(\alpha-\beta)(\alpha-1)\beta^{-1}} t^{\alpha-1} \Xi_{\alpha, \alpha} \left( \langle \xi, \xi \rangle^{\alpha \beta} t^\alpha \right).
\]
Remarks

In progress:

1. To give an explicitly definition of the fractional Gross Laplacian.

2. If the fractional Gross Laplacian is a convolution operator?

3. Resolve
   - the Caputo time fractional evolution equation
   - the Grunwald-Letnikov fractional evolution equation

4. If it possible, give a probabilistic representation.
Preliminaries:
Gross Laplacian
Power of the Gross Laplacian
Fractional Gross Laplacian
Fractional diffusion Equations

Solution of a linear stochastic differential equations
Riemann-Liouville time fractional diffusion Equations

Thank you!